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MANAGERIAL INVENTORY FORMULATIONS WITH STOCKOUT OBJECTIVES AND FISCAL CONSTRAINTS

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ABSTRACT

Most inventory formulations seek to minimize the sum of ordering costs, holding costs, and stockout costs; however, management often directs inventory policy by specifying a maximum investment level and/or a purchasing budget constraint. Within these limitations, they expect lower level managers to optimize some level of customer satisfaction, such as minimum stockouts or minimum shortages. The author has developed several cases of these "managerial" inventory formulations and has presented some computational results.

INTRODUCTION

Although inventory theory has been highly developed, practice still lags considerably behind the elegant mathematical results which are available. Many applications utilize the simple Wilson *EOQ* formula or they base inventory levels on a "number of days of supply" criterion. There are probably many impediments to the use of more sophisticated inventory theory. Many formulations require an estimate of stockout costs which are sometimes very difficult to measure. Some practical situations require operational constraints on reorder workloads, investment levels, procurement budgets and so on. This paper proposes a constrained type of inventory model that considers both stockout criteria and operational constraints in the formulation.

A very large part of the inventory literature deals with the following traditional cost model:

Minimize (holding cost + ordering cost + stockout cost)

There are literally hundreds of variations in assumptions within the above cost framework. For some problems the traditional cost framework is quite adequate, from a practical point of view. However, when stockout costs are difficult or impossible to measure, the traditional model becomes inadequate. In many cases no quantitative estimate of stockout cost can be made. The primary reason for avoiding stockouts in the first place may be due to competitive pressures and the real effect of a stockout is to alter future demand, rather than to incur any measurable "costs" in the usual sense of the word. In these cases it becomes appropriate to consider stockouts as the objective function of the model. We shall take such a point of view in this paper. The formulations considered are:

Min (a stockout measure)

(1)

Subject to: (operational constraints).

There are many different measures which can be used for stockouts, and indeed, no one measure is appropriate for all situations. Some common and appropriate expected value stockout measures are:

1. Expected total units backordered (or lost sales) per year.
2. Expected number of stockout occurrences per year.
3. Expected backordered units at any point in time.

As a matter of fact, in terms of business records that might be kept, the first criterion amounts to counting the number of units backordered (or lost) over some period of time; the second criterion corresponds to counting the number of times that a stockout occurred over a period of time regardless of how long it lasted or the number of units that were out of stock; and the third criterion corresponds to counting the backorders outstanding at say the end of each day and averaging this figure over some period of time. We will utilize all three criteria in the remainder of the paper.*

It is interesting to note that stockout measure number 1 corresponds closely to three other measures of system performance. It can be converted to a fraction of demand not filled immediately from stock by dividing by the total demand per year. In the steady state this fraction is also equal to the probability that the system is out of stock at any point in time. It is also equal to the expected fraction of the year that the system will be out of stock.† Thus measure number (1) represents other common criteria, as well.

There are a number of different operational constraints that could be employed in (1). The most common include:

(1) Cost per year (holding cost + ordering cost).
(2) Reorder workload per year.
(3) Investment level.
(4) Space available.
(5) Procurement budget.

These constraints could be interpreted in either an expected value or probability sense, but we shall concentrate on the expected value formulations. In any particular formulation, one or several of the above constraints could be appropriate depending on the business situation. The "managerial inventory problem" thus is to minimize one of the stockout measures subject to whatever operational constraints are appropriate. No stockout cost or penalty is explicitly included in any of the proposed models.

The remainder of the paper will discuss several cases of managerial inventory formulations. We will show how imputed stockout costs can be determined; we will compare optimal reorder points and order quantities for several models; we will explore the sensitivity of the basic models and we will relate our results to classical inventory theory. All the formulations will make use of existing theory to the maximum extent possible.

The literature contains a few recent examples of formulations of the type of Equation (1). Schrady and Choe [11] developed a (Q, r) model for objective type (3), multiple items, and backlogged demand. Although their model is one of the same general type, they treated different issues than those discussed here. This type of model has also been discussed by Porteus [9] and Gerson and Brown [3]. A number of other authors have treated the problem of minimizing cost subject to stockout constraints which is the

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*There are also some meaningful probability measures of stockout performance, but these will not be explored here.

†These points are more fully discussed in Hadley and Whitin [4].

reverse of the models proposed here. These references include Presutti and Trepp [10], Hadley and Whitin [4, p. 219], Fromovitz [2], Iglehart and Jaquette [7], and Herron [5].

FORMULATION

In this section we will reduce the problem focus to one particular case. We consider the continuous review, single product, backlogged demand inventory model with the probability distribution of demand over the lead time known and of the continuous type. We use policies of the (Q, r) type, where Q is the reorder quantity and r is the reorder point. When the inventory position (on hand + on order − back-ordered) drops to the level r, an amount Q will be ordered. For the traditional cost structure (holding + ordering + stockout), this model has been extensively treated. But there has been very little done for constrained inventory formulations of the type considered here.

Notation*

Q = order quantity
r = reorder point
λ = expected demand per year
μ = expected demand over the lead time
σ = standard deviation of demand over lead time
$h(x)$ = probability density of demand over lead time
$H(x)$ = complementary cumulative of $h(x)$
A = fixed cost per order placed
C = unit cost per item ordered
I = investment charge per dollar invested per year
β = specified cost limit per year
$E(r)$ = expected units backordered per replenishment period for reorder point r

The first case we will formulate is:

$$\text{Min (Expected units backordered per year)} \tag{2}$$
$$\text{Subject to: (Holding + ordering) cost per year} = \beta$$

The objective function in (2) can be written as:

$$f(Q, r) = [\lambda/Q]E(r), \tag{3}$$

where λ/Q is the expected number of replenishment periods per year and $E(r)$ is the expected number of units backordered per replenishment period. There is an assumption built in here that all backorders are filled when stock arrives. This will hold when expected backorders are low, which is often the case in practical applications. As will be discussed later, the assumption will be satisfied if Q/σ is large.

We also have by definition:

* The notation adopted follows that of Hadley and Whitin [4] for ease of reference.

$$E(r)=\int_r^{\infty}(x-r)h(x)\,dx. \tag{4}$$

The constraint in (2) is:

$$[\lambda/Q]A+IC[Q/2+r-\mu], \tag{5}$$

where $[\lambda/Q]A$ is the fixed cost of reordering per year and $IC[Q/2+r-\mu]$ is the holding cost per year based on the approximate average on-hand inventory. The approximation of on-hand inventory is also based on the previously indicated assumption that expected backorders are low. The mathematical formulation of the problem in (2) is then:

$$\begin{aligned}&\text{Min } \frac{\lambda}{Q}\int_r^{\infty}(x-r)h(x)\,dx\\ &\text{subject to: } \frac{\lambda}{Q}A+IC\left[\frac{Q}{2}+r-\mu\right]=\beta.\end{aligned} \tag{6}$$

The Lagrangian equivalent of (6) is

$$L(r, Q, \pi)=\frac{\lambda}{Q}\int_r^{\infty}(x-r)h(x)\,dx+\pi\left[\frac{\lambda}{Q}A+IC\left(\frac{Q}{2}+r-\mu\right)-\beta\right], \tag{7}$$

where π is the Lagrange multiplier. Since $L(r, Q, \pi)$ is convex in r and Q when $r \geq \mu$ [1], solving (7) is completely equivalent to solving (6). The restriction $r \geq \mu$ is consistent with the previous assumption that expected backorders are low. The restriction is almost always satisfied in applications.

The optimal values of Q and r must satisfy the following first order derivative conditions:

$$\frac{\partial L}{\partial Q}=-\frac{\lambda}{Q^2}E(r)+\pi\left[\frac{-\lambda A}{Q^2}+\frac{IC}{2}\right]=0, \tag{8}$$

$$\frac{\partial L}{\partial r}=\frac{\lambda}{Q}[-\dot{H}(r)]+\pi IC=0. \tag{9}$$

Solving Equation (8) for Q and Equation (9) for $H(r)$, we get

$$Q=\sqrt{\frac{2\lambda[E(r)+\pi A]}{\pi IC}}, \tag{10}$$

$$H(r)=\frac{\pi ICQ}{\lambda}. \tag{11}$$

The right-hand side of Equation (11) must be restricted to the interval [0, 1], since $H(r)$ is a probability.

Thus (11) is only valid for suitable parameters of a particular problem. This restriction is necessary due to our previous assumption that backorder levels are low. In general, it is not possible to solve Equations (10) and (11) for Q and r in closed form. However, for practical purposes they can be solved iteratively for the desired values of Q and r.*

Equations (10) and (11) are too complex to allow for generalized conclusions about the original problem. It will be necessary to further assume a particular type of demand density function in order to get specific results.

We will assume that the distribution of demand over the lead time is normal. In many practical applications this is not an overly restrictive assumption. To obtain a manageable analysis it is useful to further approximate the normal distribution tail with a simple exponential function. We will assume that

$$H(x) = ae^{-b\left(\frac{x-\mu}{\sigma}\right)}. \tag{12}$$

The constants a and b can be chosen to give a good fit for any particular problem. For example, if $H(r)$, the probability of a stockout during a replenishment period, should be relatively small then the values of a and b should be chosen to give a good fit for the extreme right tail of the distribution.

Figure 1 shows the standard normal distribution $H(z)$ for $1 \leq z \leq 3$ and the corresponding fit from Equation (12).† In this case the least squares value for a and b were $a = 2.88$, $b = 2.49$. The fit achieved over this range was excellent. The exponential approximation to the normal was first used for inventory work by Parker [8] and later by Presutti and Trepp [10], but different values of a and b were used than those selected here.

Using the exponential approximation, Equation (4) for $E(r)$ can be integrated to yield

$$E(r) = \frac{\sigma a}{b} e^{-\left(\frac{r-\mu}{\sigma}\right)b}. \tag{13}$$

In this case $E(r)$ and $H(r)$ both have a simple exponential form. The expression for $E(r)$ can be substituted into Equations (8) and (9) to yield the following closed form equation for Q.

$$Q = \frac{\sigma}{b} + \sqrt{\left(\frac{\sigma}{b}\right)^2 + \frac{2\lambda A}{CI}}. \tag{14}$$

Note that the equation for Q is independent of the reorder point, r, and the Langrange multiper, π. Furthermore, when σ tends to zero, Q will approach the familiar Wilson EOQ formula. This same result was obtained by Presutti and Trepp [10] for the multiple item case. Finally, it should be noted that the value of Q computed from (14) will be somewhat larger than the corresponding Wilson EOQ in all cases.

The value of r must now be chosen to satisfy the constraint of the original problem in (6).

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*See Hadley and Whitin [4].

† z is the standard normal variable, $z = \frac{x-\mu}{\sigma}$.

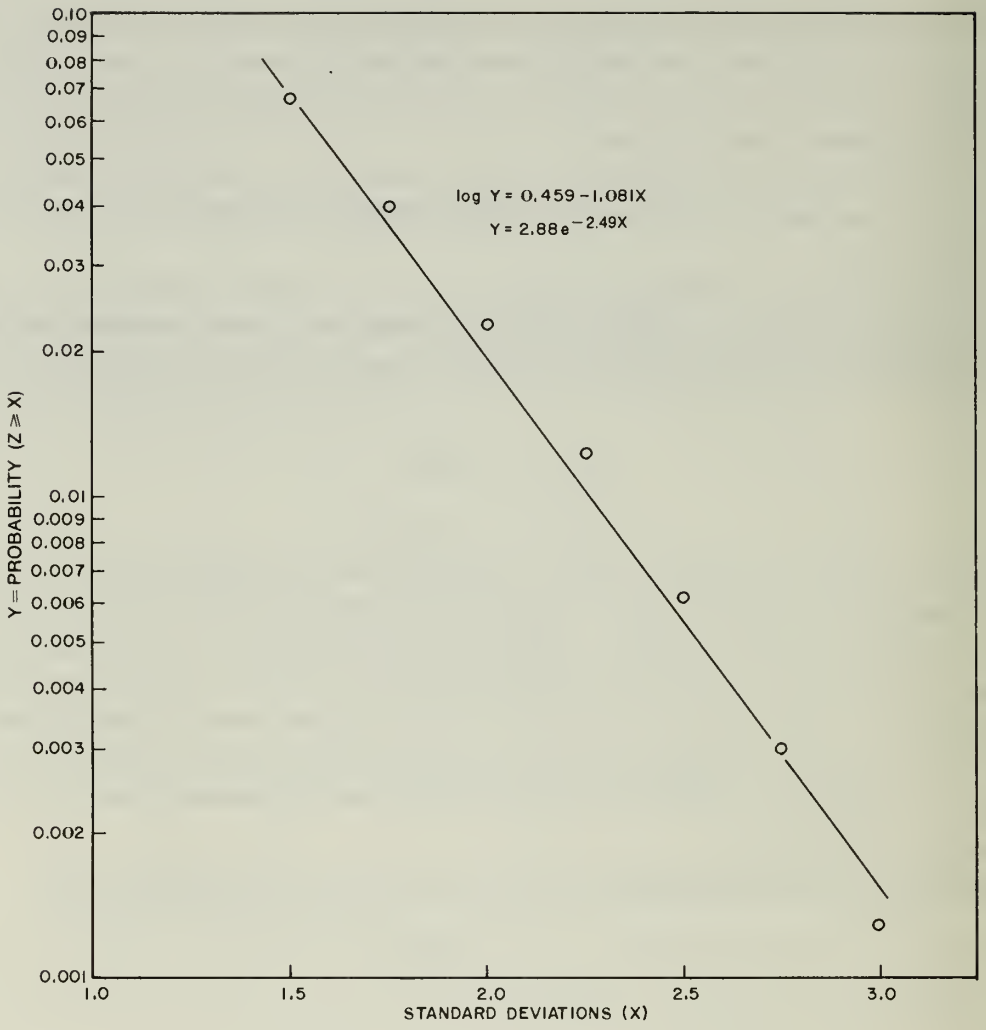

FIGURE 1. Standard normal tail approximation.

$$r = \frac{1}{IC}\left(\beta - \frac{\lambda}{Q}A\right) + \mu - \frac{Q}{2}. \tag{15}$$

The value of r will depend on Q and the other parameters of the problem. The appropriate value of the Lagrange multiplier would now be determined by the values of Q, r, and Equation (11) which must be satisfied.

To summarize, the order quantity Q can be set independently of the specified cost constraint β and the reorder point r. Thus the problem has become separable in the decision variables Q and r and the cost constraint determines only the reorder point and not the order quantity.

The formulas that have been developed can be used to aid a manager's judgment in selecting an optimal inventory policy. The relevant information is the relationship between the optimal objective function of (6) and the constraint value β. This relationship can be expressed in closed form by substituting (14) and (15) into (3) to yield the optimal value of the objective function as a function of β.

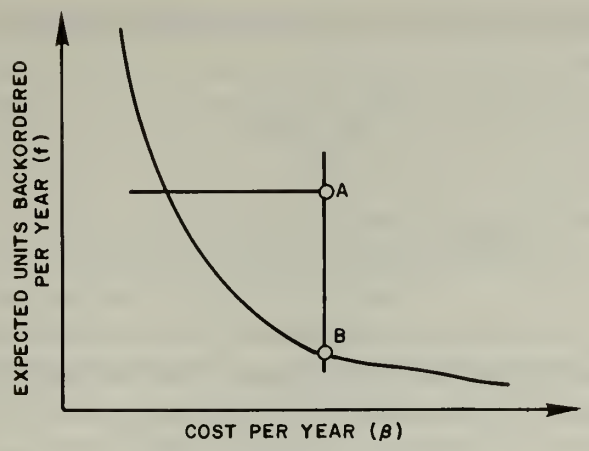

FIGURE 2. Efficient tradeoff curve.

$$f = C_1 e^{-C_2 \beta}, \tag{16}$$

where C_1 and C_2 are constants.

Figure 2 is a graph of Equation (16). This is the most efficient tradeoff curve in the sense that an optimal policy is used at all points on the curve. The curve indicates how one must tradeoff dollars (β) to gain reduced backorder levels (f). From this curve a manager could indicate his preference for increasing costs and reducing backorders. In this sense, the tradeoff is analogous to the Pareto optimum concept in Welfare Economics. This concept should be used in inventory applications when stockouts and costs are incommensurable objectives.

The curve in Figure 2 indicates an efficiency frontier; for each fixed value of β, the curve shows the minimum value of f and vice-versa. If the current system were operating at point A, it would be more efficient to move to a point on the curve and decrease both cost and backorders. For example, one possibility would be to move to point B and reduce backorders at constant cost.

Each point on the tradeoff curve corresponds to a value of the Lagrange multiplier π. The physical interpretation of π is the number of backorders incurred per dollar spent. This is the reciprocal of the usual stockout cost. Thus when a particular operating point in Figure 2 is chosen, a stockout cost is automatically imputed. These imputed costs would be very helpful in assisting managers to select a value of β for operations.

Since we have a closed form expression in (16), it is clear that f is a convex function of β. This is difficult to demonstrate for more general probability distributions. Equation (16) also allows extensive sensitivity comparisons to be made for the parameters A, λ, μ, etc. Sensitivity analysis is, of course, useful for investigating optimal policies.

OTHER STOCKOUT CRITERIA

We have discussed in some detail, one particular objective function, expected units backordered per year. In this section we will summarize some results for two other stockout measures; expected number of stockout occurrences per year, and expected backordered units at any point in time. Only the main results will be summarized for these cases, since the development is similar to the case already discussed.

For the expected number of stockout occurrences per year as the objective function, the mathematical formulation is

$$\text{Min } \frac{\lambda}{Q} H(r)$$

(17)

$$\text{subject to: } \frac{\lambda}{Q} A + IC\left[\frac{Q}{2} + r - \mu\right] = \beta.$$

The objective function is the product of the expected number of replenishments per year and the probability of a stockout on each replenishment occasion. The constraint is the same as before.

In this case we also want to use the exponential approximation for $H(x)$ from (12). The Lagrangian function is then

$$L(Q, r, \pi) = \frac{\lambda}{Q}\left[ae^{-b\left(\frac{r-\mu}{\sigma}\right)}\right] + \pi\left[\frac{\lambda}{Q} A + IC\left(\frac{Q}{2} + r - \mu\right) - \beta\right]. \tag{18}$$

The first order partial derivative conditions are

$$\frac{\partial L}{\partial Q} = -\frac{\lambda}{Q^2}\left[ae^{-b\left(\frac{r-\mu}{\sigma}\right)}\right] + \pi\left[-\frac{\lambda A}{Q^2} + \frac{IC}{2}\right] = 0, \tag{19}$$

and

$$\frac{\partial L}{\partial r} = \frac{\lambda}{Q}\left[-\frac{b}{\sigma} ae^{-b\left(\frac{r-\mu}{\sigma}\right)}\right] + \pi\,[IC] = 0. \tag{20}$$

We can solve Equation (20) for $e^{-b\left(\frac{r-\mu}{\sigma}\right)}$ and substitute this expression into (19) to eliminate r from that equation. The result is

$$-\frac{\lambda}{Q^2}\left[\frac{\pi ICQ\sigma}{\lambda b}\right] + \pi\left[-\frac{\lambda A}{Q^2} + \frac{IC}{2}\right] = 0. \tag{21}$$

This is a Quadratic equation in Q. The solution to the equation is*

$$Q = \frac{\sigma}{b} + \sqrt{\left(\frac{\sigma}{b}\right)^2 + \frac{2\lambda A}{IC}}. \tag{22}$$

This is exactly the same value of Q as that obtained for the previous expected units backordered objective function, see Equation (14). Since Q is the same, r will also be the same because r is derived from the cost constraint. It is an interesting result that the optimal policy (Q, r) is not dependent on the form of the objective function chosen; the two cases have the same optimal values of (Q, r).

*The negative root of Q has been discarded from further consideration, since we want only positive order quantities.

We note that this result could not have been obtained from the classical formulation of inventory costs, if the cost of a stockout were held constant between the two cases. Indeed, the imputed stockout costs are not the same. The stockout costs also have a different physical interpretation. In the first case the stockout cost is a cost per unit backordered and in the second case it is a cost per stockout occurrence. We would not expect the same value of π to apply for these two cases.

The third objective function of interest is the expected units backordered at any point in time. For this objective function, the number of backorders at a point in time is integrated out over all probable values to arrive at the expected number of backorders.

A technique developed by Hadley and Whitin [4, p. 192] can be used to derive an expression for the expected number of backordered units at any time. The inventory position can take on any value between r and $r+Q$. The probability that the inventory position lies between x and $x+dx$ with $r \leq x \leq r+Q$ at any random time is simply dx/Q. The probability density that x units are demanded during the leadtime with the exponential approximation is

$$h(x) = \frac{ab}{\sigma} e^{-b\left(\frac{x-\mu}{\sigma}\right)}. \tag{23}$$

Let $g(y)$ be the probability density for the quantity of backorders at any time t, it follows that

$$g(y) = \int_r^{r+Q} h(x+y)\frac{dx}{Q} = \int_r^{r+Q} \frac{ab}{\sigma Q} e^{-b\left(\frac{x+y-\mu}{\sigma}\right)} dx = \left(1-e^{-\frac{bQ}{\sigma}}\right)\frac{a}{Q} e^{-\frac{b}{\sigma}(r+y-\mu)}. \tag{24}$$

From the density of the number of backorders at any time, it is simple to integrate and get expected backorders, $B(Q, r)$

$$B(Q, r) = \int_0^{\infty} yg(y)dy = \frac{a\sigma^2}{Qb^2}\left(1-e^{-\frac{bQ}{\sigma}}\right) e^{\frac{b}{\sigma}(r-\mu)} \tag{25}$$

To obtain consistency with the previous two objective functions it is appropriate to assume that bQ/σ is large. This is equivalent to an assumption that demand during the lead time is not likely to exceed $Q+r$. This assumption will be satisfied when a low level of backorders is maintained, which is fortunately true in most practical applications. The third optimization problem is then

$$\text{Min } \frac{a\sigma^2}{b^2Q} e^{-\frac{b(r-\mu)}{\sigma}}$$

$$\text{subject to: } \frac{\lambda}{Q}A + IC\left(\frac{Q}{2}+r-\mu\right) = \beta. \tag{26}$$

From the Lagrange formulation it is easy to show that the optimal values of (Q, r) are the same for this problem, as those derived for the previous two cases. Thus we have the result that the optimal policy is independent of which of the three objective functions are used. Presutti and Trepp [10] noted that the optimal Q was invariant for a related set of problems. However, they did not obtain invariant reorder points because of the multiple item problems which they considered.

INVESTMENT CONSTRAINT

One interesting case occurs when the cost constraint in the above problems is replaced by an investment level constraint. Then we would want to minimize a stockout measure subject to a limit on the investment level only. Such a constraint is appropriate when the cost of ordering is small or fixed; then investment level is the predominant operational consideration.

The investment constraint is a special case of the problem already considered. Simply set $A = 0$ and $I = 1.0$ in the above formulations. In this case the optimal order quantity and reorder point from (14) and (15) become

$$Q = \frac{2\sigma}{b}, \qquad r = \beta/C + \mu - \frac{\sigma}{b}. \tag{27}$$

It is interesting to note that the formulas for Q and r become very simple. The order quantity depends only on σ. The reorder point formula also has a very simple form.

MULTIPLE ITEMS

Another interesting variation, and one of considerable practical importance, is the case of multiple items. The problem that we have been considering can be written for multiple items as

$$\text{Min} \sum \frac{D_i}{Q_i} e^{-k_i b}$$

$$\text{subject to: } \sum \frac{\lambda_i}{Q_i} A_i + C_i I \left(\frac{Q_i}{2} + k_i \sigma_i \right) = \beta, \tag{28}$$

where D_i is a constant and $k_i = \dfrac{r_i - \mu_i}{\sigma_i}$ is the standard normal variate. All three cases of stockout objectives are represented by (28); but different values of D_i are needed for each case. We also may assume that the D_i term will include any linear weighting factors which indicate that a stockout of one item is more serious than other items.

The Lagrangian for (28) is

$$L(Q_i, k_i, \pi) = \sum \frac{D_i}{Q_i} e^{-k_i b} + \pi \left[\sum \frac{\lambda_i}{Q_i} A_i + C_i I \left(\frac{Q_i}{2} + k_i \sigma_i \right) - \beta \right] \tag{29}$$

The first order derivative conditions are

$$\frac{\partial L}{\partial Q_i} = -\frac{D_i}{Q^2} e^{-k_i b} + \pi \left[-\frac{\lambda_i A_i}{Q_i^2} + \frac{C_i I}{2} \right] = 0 \tag{30}$$

$$\frac{\partial L}{\partial k_i} = -\frac{D_i b}{Q_i} e^{-k_i b} + \pi C_i I \sigma_i = 0. \tag{31}$$

By substituting (31) into (30) we can eliminate k_i from (30) to get

$$\frac{\pi C_i I \sigma_i}{Q_i b} + \pi \left[-\frac{\lambda_i A_i}{Q_i^2} + \frac{C_i I}{2} \right] = 0. \tag{32}$$

This is exactly the same quadratic form as the previous single-item cases, see (21). The solution then is

$$Q_i = \frac{\sigma_i}{b} + \sqrt{\left(\frac{\sigma_i}{b}\right)^2 + \frac{2\lambda_i A_i}{IC_i}} .$$

The important result here is that the Q_i obtained for the multiple item case is the same as if the order quantities had been computed one item at a time. There is no interaction between items, as far as order quantities are concerned.

There is a multi-item interaction for reorder points. This can be seen by substituting Equation (31) back into the constraint in (28).

$$\sum \left(\frac{\lambda_i}{Q_i} A_i + \frac{C_i I Q_i}{2}\right) - \frac{1}{b} \sum C_i I \sigma_i \ln \left(\frac{\pi_i C_i I Q_i \sigma_i}{D_i b}\right) = \beta . \tag{33}$$

This constraint can be solved for the value of π. The result can be substituted into Equation (31) to obtain a closed-form expression for the k_i. However, in this case the values of k_i will not be the same for all three objective functions. This is due to the fact that the items interact in determining the value of π for the constraint. In fact the equations for k_i are quite complicated functions of the parameters involved and no simple conclusions can be obtained. These particular results are closely related to the work of Presutti and Trepp [10].

One interesting result for the multiple item case is that there will be a different optimal service level for each item. The model stipulates that one value of π should be determined for all items based on the tradeoff desired between stockouts and the cost constraint. This value of π in turn determines a different value of k_i for each item and thus a different service level. It would be inappropriate to set a service level arbitrarily for each item (or a class of items) as is often done in practice. It is more appropriate to set a system wide stockout cost and proceed from there to calculate appropriate service levels that will in general depend on the cost of the item, the mean and standard deviation of demand and so on.

EXAMPLE

An example was constructed to illustrate some of the points which have been made. The following parameters were chosen:

$\lambda = 10{,}000$	$A = 100$
$\mu = 2{,}000$	$C = 10$
$\sigma = 250$	$I = 0.2$

Figure 3 shows the expected units backordered per year as a function of the cost (β). The computations were made from the exact normal equations. For a given value of π, Equations (10) and (11) were solved iteratively to arrive at optimal values of Q and r. The corresponding units backordered and cost were then plotted on Figure 3.

Figure 3 can be used to aid a manager in determining an appropriate stockout policy. The suggested procedure is to first establish a range of stockout costs to be considered. For example, suppose

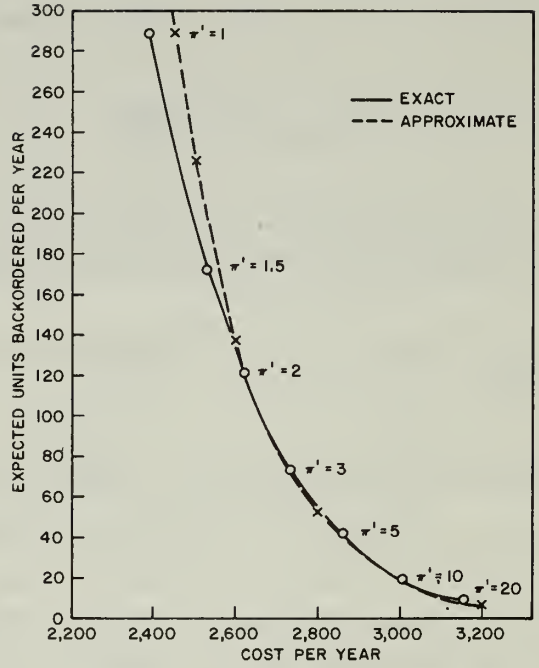

FIGURE 3

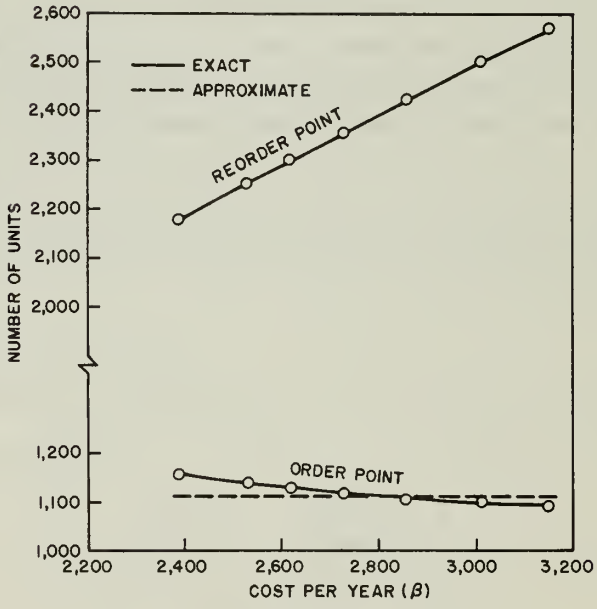

FIGURE 4

that we think the range of cost per unit backordered should run from 10 percent of the unit cost of the item to 200 percent of the unit cost or from \$1 per unit to \$20 per unit. The 10-percent extreme might represent the case where most customers are perfectly willing to wait for merchandise to arrive. The 200-percent extreme might represent the case of very impatient customers, where future demand is seriously reduced by backorders. Of course, the manager's subjective judgment is needed to select

a value between these two extremes. He can exercise his judgment by reviewing the tradeoff between backorders and costs in Figure 3. He must make a judgment on how much he would be willing to pay to reduce backorders by one additional unit. The point selected will then determine an optimal inventory policy.

Figure 4 illustrates the relationship between the optimal r, Q and the cost per year (β). As indicated, r has nearly a linear relationship and Q is nearly constant over the range of β selected. Here, we are also using the exact normal equations.

Finally, we have recomputed the quantities shown in Figures 3 and 4 for the exponential approximation to the normal. In Figure 3 quite good agreement was achieved. This was due to the close curve fit which was originally obtained from Figure 1. If a wider range of β were to be used, the approximation would not be nearly as good, unless separate approximations were used for different ranges of β. In Figure 4 a nearly perfect fit was obtained for the reorder point curve. The approximation on the order quantity curve was not quite as good, since the Q value was constant as a function of β. It appears that the approproximate method gives excellent results if it is used for the appropriate range of the normal distribution.

SUMMARY

The key results of the paper may be summarized as follows:

1. We indicated that a variety of managerial inventory formulations are possible with three different types of objective functions and several types of constraints. These formulations are important when a stockout cost cannot be objectively determined, so that managerial judgment is required to assess the tradeoff between stockouts and operational considerations.

2. For the (Q, r) backorder inventory model, we derived the optimal Q and r for an arbitrary demand distribution. These equations were too complex to draw further implications, so a simple exponential approximation to the normal tail due to Parker [8] was used in the remainder of the paper.

3. For the expected backorders per year case, we derived a closed form expression for Q and r. The formula for Q was independent of r and the cost constraint limit β. The formula for Q reduced to the Wilson EOQ, when σ tended to zero.

4. The idea of an efficiency frontier was discussed as an aid for determining the optimal tradeoff between backorders and the cost constraint. This tradeoff could be used to define the Pareto optimum between the two relevant criteria. The tradeoff curve was shown to be convex.

5. Three different stockout criteria were all shown to result in the same optimal values of Q and r. Thus the optimal inventory policy was independent of the particular form of the objective function chosen.

6. When the investment cost level is used as the operational constraint, the formulas for the optimal Q and r become very simple and easy to calculate.

7. Multiple items result in the same expression for Q_i as the single item case. Thus Q_i is independent of the multiple item cost constraint. The values of r_i may be computed in closed form, but they are different for each of the three objective functions considered. This result is closely related to [10].

8. Based on the model, the correct procedure for setting service levels is to determine an appropriate value of π for all items and then proceed to calculate separate service levels for each individual item.

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THE MIXED ORDERING POLICY IN PERIODIC REVIEW STOCHASTIC MULTI-COMMODITY INVENTORY SYSTEMS*

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ABSTRACT

In multi-commodity inventory systems with variable set-up costs, the mixed ordering policy assumes that commodities may be ordered either individually, or may be arbitrarily grouped for joint ordering. Thus, for a two-commodity system, commodity one or commodity two or commodities one and two may be ordered incurring respectively fixed order costs of K, K_1, or K_2, where max $(K_1, K_2) \leq K \leq K_1 + K_2$. This paper considers a two-commodity periodic review system. The stationary characteristics of the system are analyzed, and, for a special case, explicit solutions are obtained for the distribution of the stock levels at the beginning of the periods. In a numerical example, optimal policy variables are computed, and the mixed ordering policy is compared with individual and joint ordering policies.

INTRODUCTION

For multi-commodity systems with setup costs dyadic policies of the form "order nothing or order all commodities simultaneously" have been investigated in the literature (Ref. [1] to [14]). Other policies involving either joint orders or individual orders or no orders are realizable. Such policies will be called mixed ordering policies; these have been investigated in [2], [6], and [8] for a two-commodity single period system. In this paper, the stationary characteristics of a period review inventory system involving two commodities are analyzed for a special type of mixed ordering policy. Immediate delivery of orders and complete backlogging are assumed.

The Mixed Ordering Policy

The mixed ordering policy that has been shown in [2], [6], and [8] to be optimal for the single period two-product inventory system will be used as the ordering policy in this paper.

In the space E^2 referred to by the set of Cartesian coordinates $\underline{y} = (y_1, y_2)$, the regions o_1, o_2,

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o_{12}, $\bar{o}$, and $\bar{O}$ are constructed as follows (Figure 1):

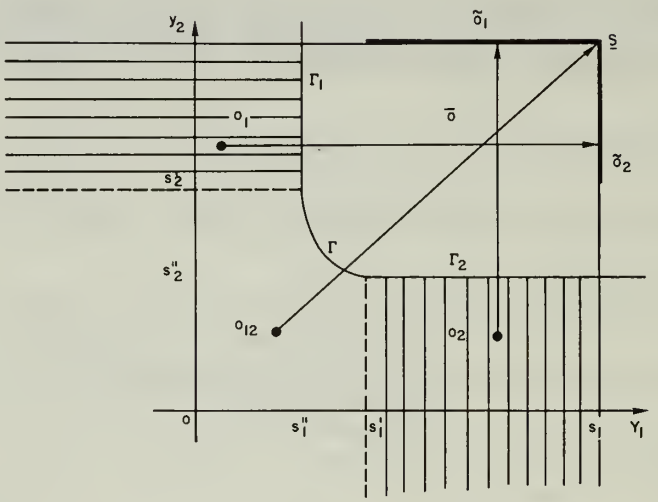

FIGURE 1. The mixed ordering policy space for a two-product inventory system.

Let

$\underline{S} = (S_1, S_2)$ be a point in E^2 with $\underline{S} > \underline{O}$;

s_1', s_2', s_1'', s_2'' be constants and $\underline{s}' = (s_1', s_2')$, $\underline{s}'' = (s_1'', s_2'')$;

$\Omega = \{\underline{y} \mid \underline{y} \leqq S\}$.

Let the function $Z\ (\underline{S} - \underline{y})$ satisfy the following properties:

a) $Z\ (\underline{S} - \underline{y})$ is defined and continuous for all $\underline{y} \in \Omega$;

b) $Z\ (\underline{S} - \underline{\eta}) \lneqq 0$ for all $\underline{\eta} \in \{\underline{\eta} \mid \underline{\eta} \in \Omega, \underline{\eta} \leqq \underline{y}, Z\ (\underline{S} - \underline{y}) = 0\}$, where the strict vector inequality is interpreted as a strict inequality for each component of the vector;

c) if $\Gamma = \{\underline{y} \mid \underline{y} \in \Omega, Z\ (\underline{S} - \underline{y}) = 0\}$ and

$\Lambda(\underline{y}) = \{\underline{\eta} \mid \underline{y} < \underline{\eta} < \underline{S}\}$, then $\Gamma \cap \Lambda(\underline{y}) = \phi$ for all $\underline{y} \in \Gamma$.

If $(s_1'', s_2') \in \Gamma$, $(s_1', s_2'') \in \Gamma$, we shall say Γ is admissible.

Let

$\Gamma_1 = \{\underline{y} \mid y_1 = s_1'', y_2 > s_2'\}$,

$\Gamma_2 = \{\underline{y} \mid y_1 > s_1', y_2 = s_2''\}$,

$o_1 = \{\underline{y} \mid y_1 \leqq s_1'', y_2 > s_2'\}$,

$o_2 = \{\underline{y} \mid y_1 > s_1', y_2 \leqq s_2''\}$,

$o_{12} = \{\underline{y} \mid \underline{y} \in \Omega, Z\ (\underline{S} - \underline{y}) > 0, \underline{y} \leqq \underline{s}'\}$, $\underline{s}' = (s_1', s_2')$,

$\bar{o} = \{\underline{y} \mid \underline{y} \in \Omega, Z\ (\underline{S} - \underline{y}) < 0, \underline{y} > \underline{s}''\}$, $\underline{s}'' = (s_1'', s_2'')$,

$\bar{O} = E^2 - (o_1 \cup o_2 \cup o_{12})$.

Let $\underline{x} = (x_1, x_2)$ be the inventory position just prior to review and $\underline{y} = (y_1, y_2)$ be the inventory position following a review then the mixed ordering policy is given as follows:

1. If $\underline{x} \in o_1$, then order product one only to bring its stock level up to $y_1 = S_1$;
2. if $\underline{x} \in o_2$, then order product two only to bring its stock level up to $y_2 = S_2$;
3. if $\underline{x} \in o_{12}$, then order both products to bring their stock level up to $\underline{y} = \underline{S}$;
4. if $\underline{x} \in \bar{o}$, then do not order.

THE STATIONARY DISTRIBUTION

The steady-state characteristics of the two-product system are studied for an ordering policy whose structure possesses the form that has been outlined. Since the stationary behavior of the system is investigated, the state space for the inventory levels will be Ω.

Let

$\underline{D}=(D_1, D_2)$ be a random vector describing the demand during a review period with distribution function $\Phi_{\underline{D}}(\underline{\xi})$, $(\underline{0} \leqq \underline{\xi} < \underline{\infty})$, and density function $\phi_{\underline{D}}(\underline{\xi})$, where $\underline{D}$ is identically and independently distributed from period to period.

$\Psi(\underline{y})=P\{\underline{Y} \leqq \underline{y}\}$ be the distribution function of the stationary distribution of $\underline{Y}$, the inventory position following a review. Then the expression for $\Psi(\underline{y})$ over the four regions over which the inventory position is defined is as follows:

$$d\Psi(\underline{y}) = \begin{cases} M \text{ if } \underline{y}=\underline{S} \\ \Psi(\underline{y})\, d\underline{y} \text{ if } \underline{y}\epsilon\tilde{o} \\ m_1(y_1)\, dy_1 \text{ if } \underline{y}\epsilon\tilde{o}_1 \\ m_2(y_2)\, dy_2 \text{ if } \underline{y}\epsilon\tilde{o}_2. \end{cases}$$

where $M=P\{\underline{Y}=S\}$ is the probability of placing a joint order and

$$\tilde{o}=\{\underline{y} \mid \underline{y}\epsilon\bar{o},\ \underline{y}<\underline{S}\}$$
$$\tilde{o}_1=\{\underline{y} \mid s_1' < y_1 < S_1,\ y_2=S_2\}$$
$$\tilde{o}_2=\{\underline{y} \mid y_1=S_1,\ s_2' < y_2 < S_2\}.$$

The functions $\Psi(\underline{y})$, $m_1(y_1)$ and $m_2(y_2)$ satisfy the following system of simultaneous integral equations:

(1) $$\Psi(\underline{y})=M\phi(\underline{S}-\underline{y})+\int_{\max(s_1', y_1)}^{S_2} m_2(w_2)\phi(S_1-y_1,\ w_2-y_2)\, dw_2$$
$$+\int_{\max(s_1', y_1)}^{S_1} m_1(w_1)\phi(w_1-y_1,\ S_2-y_2)\, dw_1 + \int_{y_2}^{S_2}\int_{y_1}^{S_1} \Psi(\underline{w})\, d\underline{w}\phi(\underline{w}-\underline{y}),\ \underline{y}\epsilon\tilde{o};$$

(2) $$m_1(y_1) = \int_{-\infty}^{s_2''}\Big[M\phi(S_1-y_1, S_2-x_2) + \int_{s_2'}^{S_2} m_2(w_2)dw_2\phi(S_1-y_1, w_2-x_2)$$
$$+\int_{y_1}^{S_1} m_1(w_1)dw_1\phi(w_1-y_1, S_2-x_2) + \int_{s_2''}^{S_2}\int_{y_1}^{S_1}\psi(\underline{w})d\underline{w}\phi(w_1-y_1, w_2-x_2)\Big]\, dx_2,\ \underline{y}\epsilon\tilde{o}_1;$$

(3) $$m_2(y_2) = \int_{-\infty}^{s_1''}\Big[M\phi(S_1-x_1, S_2-y_1) + \int_{s_1'}^{S_1} m_1(w_1)dw_1\phi(w_1-x_1, S_2-y_2)$$
$$+\int_{y_2}^{S_2} m_2(w_2)dw_2\phi(S_1-x_1, w_2-y_2) + \int_{y_2}^{S_2}\int_{s_1''}^{S_1}\psi(\underline{w})d\underline{w}\phi(w_1-x_1, w_2-y_2)\Big]\, dx_1,\ \underline{y}\epsilon\tilde{o}_2.$$

The limits of integration are valid as long as Γ is admissible.

Unfortunately, solution of the above equations is not possible without knowing the explicit forms of $\phi_{\underline{D}}(\underline{\xi})$ and $Z(\underline{S}-\underline{y})$.

Two-Product System With Uniform Demand Distribution

General mixed ordering policy

The following assumptions are made for solving (1), (2), and (3). Let the class of admissible arcs Γ be defined by

$$Z(\underline{S}-\underline{y}) = \max\,(s_1''-y_1,\, s_2''-y_2) = 0;\; s_1'' \leqslant y_1 \leqslant S_1,\; s_2'' \leqslant y_2 \leqslant S_2,$$

and let

$$\underline{s}' = \underline{s}''$$

$$\phi_{\underline{D}}(\underline{\xi}) = \phi_{D_1}(\xi_1)\phi_{D_2}(\xi_2),$$

where

$$\phi_{D_1}(\xi_1) = \frac{1}{a},\; 0<\xi_1<a \qquad \text{with } a \geqslant Q_1 = S_1 - s_1''$$

$$\phi_{D_2}(\xi_2) = \frac{1}{b},\; 0<\xi_2<b \qquad \text{with } b \geqslant Q_2 = S_2 - s_2''$$

then (1), (2), and (3) become, respectively,

$$\psi(\underline{y}) = \frac{M}{ab} + \frac{1}{ab}\int_{y_2}^{S_2} m_2(w_2)dw_2 + \frac{1}{ab}\int_{y_1}^{S_1} m_1(w_1)dw_1 + \frac{1}{ab}\int_{y_2}^{S_2}\int_{y_1}^{S_1} \psi(\underline{w})d\underline{w},\; \underline{y}\epsilon\tilde{o}; \tag{4}$$

$$m(y_1) = \frac{s_2''-S_2+b}{ab}\left[M + \int_{y_1}^{S_1} m_1(w_1)dw_1 + \int_{s_2''}^{S_2} m_2(w_2)dw_2 + \int_{s_2}^{S_2}\int_{y_1}^{S_1}\psi(\underline{w})d\underline{w}\right]$$

$$+ \frac{1}{ab}\int_{s_2''-b}^{S_2-b}\int_{s_2''}^{x_2+b}\int_{y_1}^{S_1}\psi(\underline{w})d\underline{w}dx_2 + \frac{1}{ab}\int_{s_2''-b}^{S_2-b}\int_{s_2''}^{x_2+b} m_2(w_2)dw_2dx_2,\; \underline{y}\epsilon\tilde{o}_1.$$

$$m_2(y_2) = \frac{s_1''-S_1+a}{ab}\left[M + \int_{y_2}^{S_2} m_2(w_2)dw_2 + \int_{s_1''}^{S_1} m_1(w_1)dw_1 + \int_{y_2}^{S_1}\int_{s_1''}^{S_1}\psi(\underline{w})d\underline{w}\right]$$

$$+ \frac{1}{ab}\int_{s_1''-b}^{S_1-b}\int_{y_2}^{S_2}\int_{s_1''}^{x_1-a}\psi(\underline{w})d\underline{w}dx_1 + \frac{1}{ab}\int_{s_1''}^{S_1-a}\int_{s_1''}^{x_1+a} m_1(w_1)dw_1dx_1,\; \underline{y}\epsilon\tilde{o}_2.$$

It is easy to see that (4) reduces to

$$\frac{\partial^2\psi}{\partial y_2\partial y_1} = \frac{1}{ab}\psi,$$

with boundary conditions

$$\psi(S_1, y_2) = \frac{M}{ab} + \frac{1}{ab}\int_{y_2}^{S_2} m_2(w_2)dw_2,$$

$$\psi(y_1, S_2) = \frac{M}{ab} + \frac{1}{ab}\int_{y_1}^{S_1} m_1(w_1)dw_1,$$

$$\psi(S_1, S_2) = \frac{M}{ab}.$$

Assuming a solution of the form

$$\psi(y_1, y_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(S_1 - y_1)^m(S_2 - y_2)^n, \tag{5}$$

with

$$A_{mn} \geq 0$$

yields

$$\frac{\partial^2\psi}{\partial y_2 \partial y_1} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m \cdot n A_{mn}(S_1 - y_1)^{m-1}(S_2 - y_2)^{n-1}$$
$$= \frac{1}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}(S_1 - y_1)^m(S_2 - y_2)^n.$$

By equating the coefficients of equal power of $(S_1 - y_1)^m (S_2 - y_2)^n$, the following difference equation is obtained:

$$m \cdot n A_{mn} = \frac{1}{ab} A_{m-1,n-1}.$$

Substituting in (5) yields

$$\psi(y_1, y_2) = A_{oo}\sum_{i=0}^{\infty}\left(\frac{1}{i!}\right)^2\left(\frac{1}{ab}\right)^i (S_1 - y_1)^i(S_2 - y_2)^i \tag{6}$$
$$+ \sum_{n=1}^{\infty} A_{on}\sum_{m=0}^{\infty}\frac{n!}{m!(m+n)!}\left(\frac{1}{ab}\right)^m (S_1 - y_1)^m(S_2 - y_2)^{m+n}$$
$$+ \sum_{m=1}^{\infty} A_{mo}\sum_{n=0}^{\infty}\frac{m!}{(m+n)!n!}\left(\frac{1}{ab}\right)^n (S_1 - y_1)^{m+1}(S_2 - y_2)^n.$$

Using the boundary conditions, we obtain

$$A_{oo} = \frac{M}{ab}$$

$$A_{oo}+\sum_{n=1}^{\infty} A_{on}(S_2-y_2)^n=\frac{M}{ab}+\frac{1}{ab}\int_{y_2}^{S_2} m_2(w_2)dw_2$$

$$A_{oo}+\sum_{m=1}^{\infty} A_{mo}(S_1-y_1)^m=\frac{M}{ab}+\frac{1}{ab}\int_{y_1}^{S_1} m_1(w_1)dw_1.$$

These conditions imply that

$$m_1(y_1)=\sum_{r=0}^{\infty} A_r(S_1-y_1)^r,$$

$$m_2(y_2)=\sum_{r=0}^{\infty} B_r(S_2-y_2)^r.$$

It is easy to see that

$$\frac{\partial m_1}{\partial y_1}=-\frac{s_2''-S_2+b}{ab}\left[m_1+\int_{s_2''}^{S_2}\psi(y_1,w_2)dw_2\right]-\frac{1}{ab}\int_{s_1''-b}^{S_1-b}\int_{s_2''}^{x_2+b}\psi(y_1,w_2)dw_2dx_2,$$

$$\frac{\partial m_2}{\partial y_2}=-\frac{s_1''-S_1+a}{ab}\left[m_2+\int_{s_1''}^{S_1}\psi(w_1,y_2)dw_1\right]-\frac{1}{ab}\int_{s_1''-a}^{S_1-a}\int_{s_1''}^{x_1-x}\psi(w_1,y_2)dw_1dx_1,$$

with boundary conditions

$$m_1(S_1)=\frac{s_2''-S_2+b}{ab}\left[M+\int_{s_2''}^{S_2} m_2(w_2)dw_2\right]+\frac{1}{ab}\int_{s_2''-b}^{S_2-b}\int_{s_2''}^{x_2+b} m_2(w_2)dw_2dx_2,$$

$$m_2(S_2)=\frac{s_1''-S_1+a}{ab}\left[M+\int_{s_1''}^{S_1} m_1(w_1)dw_1\right]+\frac{1}{ab}\int_{s_1''-a}^{S_1-a}\int_{s_1''}^{x_1+a} m_1(w_1)dw_1dx_1.$$

Solutions to these differential equations yield expressions that are not solvable without some further assumptions into the form of the solutions of $m_1(y_1)$ and $m_2(y_2)$.

Simplest mixed ordering policy

To gain some further knowledge of the solutions, assume that $s_1'=S_2$ which eliminates $m_2(y_2)$ from the problem, as can be seen from (4). This implies that product 1 can be ordered only if product 2 is ordered. Thus, (1) and (2) become, respectively,

$$\psi(y_1,y_2)=\frac{M}{ab}+\frac{1}{ab}\int_{y_1}^{S_1} m_1(w_1)dw_1+\frac{1}{ab}\int_{y_2}^{S_2}\int_{y_1}^{S_1}\psi(\underline{w})d\underline{w},$$

$$m_1(y_1)=\frac{s_2''-S_2+b}{ab}\left[M+\int_{y_1}^{S_1} m_1(w_1)dw_1+\int_{s_2''}^{S_2}\int_{y_1}^{S_1}\psi(\underline{w})d\underline{w}\right]+\frac{1}{ab}\int_{s_2''-b}^{S_2+b}\int_{s_2''}^{x_2+b}\int_{y_1}^{S_1}\psi(\underline{w})d\underline{w}dx_2. \tag{7}$$

'his results in the partial differential equation

$$\frac{\partial^2\psi}{\partial y_2 \partial y_1} = \frac{1}{ab}\psi$$

vith the boundary conditions

$$\psi(S_1, y_1) = \frac{M}{ab}$$

$$\psi(y_1, S_2) = \frac{M}{ab} + \frac{1}{ab}\int_{y_1}^{S_1} m_1(w_1)dw_1$$

$$\psi(S_1, S_2) = \frac{M}{ab}.$$

By letting (5) be the form of the solution, (6) is obtained. Using the above boundary condition, we obtain

$$A_{oo} + \sum_{n=1}^{\infty} A_{on}(S_2 - y_2)^n = \frac{M}{ab},$$

8) $$A_{oo} + \sum_{m=1}^{\infty} A_{mo}(S_1 - y_1)^m = \frac{M}{ab} + \frac{1}{ab}\int_{y_1}^{S_1} m_1(w_1)dw_1,$$

$$A_{00} = \frac{M}{ab}.$$

Therefore

$$A_{on} = 0, \qquad n = 1, 2, \ldots$$

and

$$\psi(y_1, y_2) = A_{oo}\sum_{i=0}^{\infty}\left(\frac{1}{i!}\right)^2\left(\frac{1}{ab}\right)^i (S_1 - y_1)^i(S_2 - y_2)^i$$

$$+ \sum_{m=1}^{\infty} A_{mo}\sum_{n=0}^{\infty}\frac{m!}{(m+n)!n!}\left(\frac{1}{ab}\right)^n (S_1 - y_1)^{m+n}(S_2 - y_2)^n$$

$$= \sum_{j=0}^{\infty} A_j \sum_{k=0}^{\infty}\frac{j!}{(j+k)!k!}\left(\frac{1}{ab}\right)^k (S_1 - y_1)^{j+k}(S_2 - y_2)^k.$$

The following differential equation obtained by differentiating (7) will be used to determine the $A_j, j = 1, 2, \ldots$, coefficients

$$\frac{\partial m_1}{\partial y_1} = -\frac{s_2'' - S_2 + b}{ab}\left[m_1 + \int_{s_2''}^{S_2} \psi(y_1, w_2)\,dw_2\right] - \frac{1}{ab}\int_{s_2''-b}^{S_2-b}\int_{s_2''}^{x_2+b} \psi(y_1, w_2)\,dw_2 dx_2$$

with boundary condition

$$m_1(S_1) = \frac{s_2'' - S_2 + b}{ab} M.$$

From (8) it is seen that

$$m_1(y_1) = ab \sum_{j=1}^{\infty} jA_j(S_1 - y_1)^{j-1}.$$

Substituting

$$Q_2 = S_2 - s_2''$$

yields

$$\frac{\partial m_1}{\partial y_1} = -ab \sum_{j=2}^{\infty} j(j-1)A_j(S_1 - y_1)^{j-2}$$

$$\equiv -\frac{b - Q_2}{ab} ab \sum_{j=1}^{\infty} jA_j(S_1 - y_1)^{j-1}$$

$$-\frac{b - Q_2}{ab} \sum_{j=0}^{\infty} A_j \sum_{k=0}^{\infty} \frac{j!}{(j+k)!(k+1)!}\left(\frac{1}{ab}\right)^k (S_1 - y_1)^{j+k} Q_2^{k+1}$$

$$-\frac{1}{ab} \sum_{j=0}^{\infty} A_j \sum_{k=0}^{\infty} \frac{j!}{(j+k)!(k+1)!}\left(\frac{1}{ab}\right)^k (S_1 - y_1)^{j+k}\left(Q_2^{k+2} - \frac{Q_2^{k+2}}{k+2}\right)$$

or

$$\sum_{j=0}^{\infty} j(j+1)A_j(S_1 - y_1)^{j+2} = \frac{b - Q_2}{ab} \sum_{j=1}^{\infty} jA_j(S_1 - y_1)^{j+1}$$

$$+ \sum_{j=0}^{\infty} A_j \sum_{k=0}^{\infty} \frac{j!}{(j+k)!(k+2)!}\left(\frac{1}{ab}\right)^{k+2} (S_1 - y_1)^{j+k} Q_2^{k+1}[(k+2)b - Q_2].$$

From the boundary conditions

$$A_0 = \frac{M}{ab}$$

$$A_1 = \frac{b - Q_2}{(ab)^2} M$$

and by equating the coefficients of equal power of $(S_1 - y_1)^j$ the remaining A_j coefficients $(j = 2, 3, \ldots)$ are found to be

$$A_j = \frac{M}{(ab)^{j+1}} \frac{1}{j!} \sum_{i=0}^{j} (-1)^i \frac{b^{j-i} Q_2^i}{i!},$$

herefore

$$\psi(y_1, y_2) = \sum_{j=0}^{\infty} A_j \sum_{k=0}^{\infty} \frac{j!}{(j+k)!k!} \left(\frac{1}{ab}\right)^k (S_1 - y_1)^{j+k} (S_2 - y_2)^k$$

r

$$\psi(y_1, y_2) = \sum_{j=o}^{\infty} \frac{M}{ab} \sum_{i=0}^{j} \sum_{k=0}^{\infty} (-1)^i \frac{1}{j!} \frac{1}{b^j} \frac{b^{j-i} Q_2^i j!}{i!(j+k)!k!} \left(\frac{S_1 - y_1}{a}\right)^{j+k} \left(\frac{S_2 - y_2}{b}\right)^k.$$

et

$$d = \frac{Q_2}{b},$$

$$u = \frac{S_1 - y_1}{a},$$

$$v = \frac{S_2 - y_2}{b},$$

$$g(u, v) = \psi(S_1 - au, S_2 - bv)$$

$$= \frac{M}{ab} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \sum_{k=0}^{\infty} (-d) \frac{1}{i!(j+k)!k!} u^{j+k} v^k,$$

nd let $\bar{g}(s, v)$ be the Laplace transform of $g(u, v)$ with respect to u for all $u \geq 0$; thus we obtain

$$\bar{g}(s, v) = \frac{M}{ab} \sum_{r=0}^{\infty} \frac{(v-d)^r}{r!} \frac{1}{s^r} \frac{1}{s-1}.$$

Therefore

9) $$g(u, v) = \frac{M}{ab} \left\{ e^u - \sqrt{d-v} \int_0^u \frac{e^t}{\sqrt{u-t}} J_1 \left[2\sqrt{(d-v)\ (u-t)}\right] dt \right\},$$

where $J_1(\cdot)$ is the Bessel function of order 1. This expression can alternately be written as

$$g(u, v) = \frac{M}{ab} \sum_{m,n=0}^{\infty} \frac{1}{(m+n)!n!} u^{m+n} (v-d)^n,$$

which is a confluent hypergeometric function.

The expression for $m_1(y_1)$ is derived in the same manner and the following is obtained:

10) $$g(u) = m_1(S_1 - au) = \frac{M}{a} \left\{ e^u - de^u + d \int_0^u \frac{e^t}{u-t} J_2 \left[2\sqrt{d(u-t)}\right] dt \right\}.$$

If $s_2''=S_2$, then a one-product inventory problem is obtained, and we have

$$g(u)=\frac{M}{a}e^{u},$$

since $d=0$.

THE OPTIMIZATION PROBLEM

With a knowledge of the stationary distribution of the inventory position following a review, the optimal ordering rules can be derived. Assume that the objective function is the total expected cost per period expressed as the sum of the expectations of the following:

1. the procurement costs,
2. the holding costs measured at the end of a period,
3. the shortage costs measured at the end of a period.

For product $i(i=1, 2)$ let

c_i be the purchasing cost per unit,

h_i be the holding cost per unit,

p_i be the shortage cost per unit,

X_i be the inventory level at the end of the period,

B_i be the quantity backlogged at the end of the period,

C be the total cost per period.

Thus the expected total cost per period [6] is

$$E[C]=KM+\sum_{i=1}^{2}\left\{K_{3-i}\int_{s_i''}^{S_i}m_i(y_i)dy_i+c_iE[D_i]+h_iE[X_i]+(h_i+p_i)E[B_i]\right\}$$

Because the terms $c_iE[D_i]$ will not affect the selection of the optimal ordering rules, it is convenient to define the function

$$\bar{E}=KM+\sum_{i=1}^{2}\left\{K_{3-i}\int_{s_i''}^{S_i}m_i(y_i)dy_i+h_iE[X_i]+(h_i+p+p_i)E[B_i]\right\} \tag{11}$$

The optimization problem can then be defined as: determine the optimal values of S_i and $s_i''(i=1, 2)$ so as to minimize $\bar{E}$.

Two-Product System with Uniform Demand Distribution

The stationary distribution defined by (9) and (10) will be used to derive $E[X_i]$ and $E[B_i]$ of (11).

In the following derivations when the change of variable

$$u_i=S_i-y_i, \qquad i=1, 2$$

performed, the policy space in the y_1, y_2-plane is mapped into the first quadrant of the u_1, u_2-plane. et $R, R_1, R_2, \Gamma_1', \Gamma_2'$ refer, respectively, to the mapping of $\tilde{o}, \tilde{o}_1, \tilde{o}_2, \Gamma, \Gamma_1$ and Γ_2 under this transforma- on. From (9) and (10)

$$\psi(y) = g\left(\frac{S_1 - y_1}{a}, \frac{S_2 - y_2}{b}\right), y \epsilon \tilde{o}$$

$$m_1(y_1) = g\left(\frac{S_1 - y_1|}{a}\right), y \epsilon \tilde{o}_1.$$

et $u_i = S_i - y_i (i = 1, 2)$ and define

$$\tau(\underline{u}) = \frac{1}{ab}\left\{ \exp\left(\frac{u_1}{a}\right) - \sqrt{d - \frac{u_2}{b}} \int_o^{u_1/a} \frac{\exp(t)}{\sqrt{\frac{u_1}{a} - t}} J_1\left[2\sqrt{\left(d - \frac{u_2}{b}\right)\left(\frac{u_1}{a} - t\right)}\right] dt \right\}$$

$$\tau_1(u_1) = \frac{1}{a}\left\{ (1-d) \exp\left(\frac{u_1}{a}\right) + d \int_o^{u_1/a} \frac{\exp(t)}{\frac{u_1}{a} - t} J_2\left[2\sqrt{d\left(\frac{u_1}{a} - t\right)}\right] dt \right\},$$

ıen

2) $$\psi(\underline{y}) = M\tau(\underline{S} - \underline{y}), \quad \underline{y} \epsilon \tilde{o}$$

3) $$m_1(y_1) = M\tau_1(S_1 - y_1), \quad \underline{y} \epsilon \tilde{o}_1.$$

he normalizing condition is

4) $$M + \int_{\tilde{o}_1} m_1(y_1) dy_1 + \int\int_{\tilde{o}} \psi(\underline{y}) d\underline{y} = 1 .$$

efine $$\Delta = \int_{\tilde{o}_1} \tau_1(S_1 - y_1) dy_1 + \int\int_{\tilde{o}} \tau(\underline{S} - \underline{y}) d\underline{y} ,$$

hus

5) $$M = \frac{1}{1 + \Delta} .$$

Let $u_i = S_i - y_i (i = 1, 2)$, then

6) $$\Delta = \int_{R_1} \tau_1(u_1) du_1 + \int\int_R \tau(\underline{u}) d\underline{u} .$$

The expression for $E[X_1]$ is

$$E[X_1]=M\int_o^a (S_1-\xi_1)\frac{1}{a}\,d\xi_1+\int_{\tilde{o}_1}\int_o^a (y_1-\xi_1)\frac{1}{a}\,m_1(y_1)\,d\xi_1 dy_1+\int\int_{\tilde{o}}\int_o^a (y_1-\xi_1)\frac{1}{a}\,\psi(\underline{y})\,d\xi_1 d\underline{y}$$

or

$$E[X_1]=M(S_1-\tfrac{1}{2}a)+\int_{\tilde{o}_1} y_1 m_1(y_1)dy_1-\tfrac{1}{2}a\int_{\tilde{o}_1} m_1(y_1)dy_1+\int\int_{\tilde{o}} y_1\psi(\underline{y})d\underline{y}-\tfrac{1}{2}a\int\int_{\tilde{o}}\psi(\underline{y})d\underline{y}.$$

By letting $u_i=S_i-y_i(i=1, 2)$ and using (12), (13), (14), (15), and (16), we obtain

$$E[X_1]=S_1-\tfrac{1}{2}a-\frac{N_1}{1+\Delta}, \tag{17}$$

where

$$N_1=\int_{R_1} u_1\tau_1(u_1)du_1+\int\int_R u_1\tau(\underline{y})d\underline{y}.$$

The expression for $E[X_2]$ is

$$E[X_2]=M\int_o^b (S_2-\xi_2)\frac{1}{b}\,d\xi_2+\int_{\tilde{o}_1}\int_o^b (S_2-\xi_2)\frac{1}{b}\,m_1(y_1)d\xi_2 dy_1+\int\int_{\tilde{o}}\int_o^b (y_2-\xi_2)\frac{1}{b}\,\psi(\underline{y})d\xi_2 d\underline{y}$$

$$=M(S_2-\tfrac{1}{2}b)+(S_2-\tfrac{1}{2}b)\int_{\tilde{o}_1} m_1(y_1)dy_1+\int\int_{\tilde{o}} y_2\psi(\underline{y})d\underline{y}-\tfrac{1}{2}b\int\int_{\tilde{o}}\psi(\underline{y})d\underline{y}.$$

By letting $u_i=S_i-y_i(i=1, 2)$ and using (12), (13), (14), (15) and (16), we obtain

$$E[X_2]=S_2-\tfrac{1}{2}b-\frac{N_2}{1+\Delta}, \tag{19}$$

where

$$N_2=\int\int_R u_2\tau(\underline{u})d\underline{u}. \tag{20}$$

The expression for $E[B_1]$, under the assumptions that $S_1\leqslant a$ and $s_1''\geqslant 0$, is

$$E[B_1]=M\int_{S_1}^a (\xi_1-S_1)\frac{1}{a}\,d\xi_1+\int_{\tilde{o}_1}\int_{y_1}^a (\xi_1-y_1)\frac{1}{a}\,m_1(y_1)d\xi_1 dy_1+\int\int_{\tilde{o}}\int_{y_1}^a (\xi_1-y_1)\frac{1}{a}\,\psi(\underline{y})d\xi_1 d\underline{y}$$

$$=M\left(\tfrac{1}{2}a-S_1+\frac{S_1^2}{2a}\right)+\int_{\tilde{o}_1}\left(\tfrac{1}{2}a-y_1-\frac{y_1^2}{2a}\right)m_1(y_1)dy_1+\int\int_{\tilde{o}}\left(\tfrac{1}{2}a-y_1+\frac{y_1^2}{2a}\right)\psi(\underline{y})d\underline{y}.$$

y letting $u_i = S_i - y_i (i = 1, 2)$ and using (12), (13), (14), (15), (16), and (18), we obtain

$$E[B_1] = \frac{(a-S_1)^2}{2a} + \frac{N_1(a-S_1)}{a(1+\Delta)} + \frac{M_1}{2a(1+\Delta)}, \tag{21}$$

here

$$M_1 = \int_{R_1} u_1^2 \tau_1(u_1) du_1 + \int\int_R u_1^2 \tau(\underline{u}) d\underline{u}. \tag{22}$$

The expression for $E[B_2]$, under the assumptions that $S_2 \leq b$ and $s_2'' \geq 0$, is

$$E[B_2] = M \int_{S_2}^{b} (\xi_2 - S_2) \frac{1}{b} d\xi_2 + \int_{\tilde{o}_1} \int_{S_2}^{b} (\xi_2 - S_2) \frac{1}{b} m_1(y_1) d\xi_2 dy_1 + \int\int_{\tilde{o}} \int_{y_2}^{b} (\xi_2 - y_2) \frac{1}{b} \psi(\underline{y}) d\xi_2 d\underline{y}$$

$$= M\left(\tfrac{1}{2}b - S_2 + \frac{S_2^2}{b}\right) + \int_{\tilde{o}_1} \left(\tfrac{1}{2}b - S_2 + \frac{S_2^2}{b}\right) m_1(y_1) dy_1 + \int\int_{\tilde{o}} \left(\tfrac{1}{2}b - y_2 + \frac{y^2}{b}\right) \psi(\underline{y}) d\underline{y}.$$

By letting $u_i = S_i - y_i (i = 1, 2)$ and using (12), (13), (14), (15), (16), and (20), we obtain

$$E[B_2] = \frac{(b-S_2)^2}{2b} + \frac{N_2(b-S_2)}{b(1+\Delta)} + \frac{M_2}{2b(1+\Delta)}, \tag{23}$$

where

$$M_2 = \int\int_R u_2^2 \tau(\underline{u}) d\underline{u}. \tag{24}$$

Substituting (15), (17), (19), (21) and (23) into (11) and letting $u_i = S_i - y_i$ $(i = 1, 2)$ yields

$$\overline{E} = \tfrac{1}{2}(p_1 a + p_2 b) - (p_1 S_1 + p_2 S_2) + \tfrac{1}{2}\left[(h_1 + p_1) \frac{S_1^2}{a} + (h_2 + p_2) \frac{S_2^2}{b} \right] \tag{25}$$

$$+ \frac{1}{1+\Delta} \left\{ K + K_2 \int_{R_1} \tau_1(u_1) du_1 + (p_1 N_1 + p_2 N_2) - \left[(h_1 + p_1) \frac{S_1 N_1}{a} + (h_2 + p_2) \frac{S_2 N_2}{b} \right] \right.$$

$$\left. + \tfrac{1}{2}\left[(h_1 + p_1) \frac{M_1}{a} + (h_2 + p_2) \frac{M_2}{b} \right] \right\}.$$

Each of the expressions $\int_{R_1} \tau_1(u_1) du_1$, N_1, N_2, M_2, and M_2 is a function of Q_1 and Q_2 and is independent of S_1 and S_2. By use of calculus and using the assumption that S_1 and S_2 are continuous variables, we find the optimal values of S_1, say S_1^*, and S_2, say S_2^*, are

$$S_i^* = \frac{p_i a}{h_i + p_i} + \frac{N_i}{1+\Delta} \tag{26}$$

$$S_2^* = \frac{p_2 b}{h_2 + p_2} + \frac{N_2}{1+\Delta}. \tag{27}$$

Substituting (26) and (27) into (25) yields

$$\bar{E} = \tfrac{1}{2}(p_1 a + p_2 b) + \frac{1}{1+\Delta}\left\{ K + K_2 \int_{R_1} \tau_1(u_1)du_1 + (p_1 N_1 + p_2 N_2) + \tfrac{1}{2}\left[(h_1 + p_1)\frac{M_1}{a} + (h_2 + p_2)\frac{M_2}{b} \right] \right\} - \frac{h_1 + p_1}{2a}\left[\frac{p_1 a}{h_1 + p_1} + \frac{N_1}{1+\Delta} \right]^2 - \frac{h_2 + p_2}{2b}\left[\frac{p_2 b}{h_2 + p_2} + \frac{N_2}{1+\Delta} \right]^2. \tag{28}$$

Thus to evaluate $\bar{E}$ we need to determine the expression Δ, N_1, N_2, M_1, M_2 and $\int_{R_1} \tau_1(u_1)du_1$ in terms of known constants and Q_1 and Q_2.

The expression for Δ is given by (16). The single integral is

$$\int_{R_1} \tau_1(u_1)du_1 = \int_o^{Q_1} \frac{1}{a}\left\{ \left(1 - \frac{Q_2}{b}\right) \exp\left(\frac{u_1}{a}\right) + \frac{Q_2}{b} \int_o^{u_1/a} \frac{\exp(t)}{\frac{u_1}{a} - t} J_2\left[2\sqrt{\frac{Q_2}{b}\left(\frac{u_1}{a} - t\right)}\, dt \right] \right\} du_1$$

$$= \left(1 - \frac{Q_2}{b}\right)\left[\exp\left(\frac{b_1}{a}\right) - 1 \right] + \int_0^{Q_2} \int_0^{u_1/a} \frac{\exp(t)}{a} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+2)!\,r!} \left(\frac{Q_2}{b}\right)^{r+2} \left(\frac{u_1}{a} - t\right)^r dt du_1. \tag{29}$$

The double integral is

$$\iint_R \tau(\underline{u})d\underline{u} = \int_o^{Q_1} \int_o^{Q_2} \frac{1}{ab}\left\{ \exp\left(\frac{u_1}{a}\right) - \sqrt{\frac{Q_2 - u_2}{b}} \int_0^{u_1/a} \frac{\exp(t)}{\sqrt{\frac{u_1}{a} - t}} J_1\left[2\sqrt{\left(\frac{Q_2 - u_2}{b}\right)\left(\frac{u_1}{a} - t\right)} \right] dt \right\} d\underline{u}$$

$$= \frac{Q_2}{b}\left[\exp\left(\frac{Q_1}{a}\right) - 1 \right] - \int_o^{Q_1} \int_o^{Q_2} \int_o^{u_1/a} \frac{\exp(t)}{ab} \sum_{r=0}^{\infty} \frac{(+1)^r}{(r+1)!\,r!} \left(\frac{Q_2 - u_2}{b}\right)^{r+1} \left(\frac{u_1}{a} - t\right)^r dt du_2 du_1.$$

Changing the order of integration for u_2 and t, then showing the series converges uniformly for $u_2 \epsilon [0, Q_2]$, yields

$$\iint_R \tau(\underline{u})d\underline{u} = \frac{Q_2}{b}\left[\exp\left(\frac{Q_1}{a}\right) - 1 \right] - \int_o^{Q_1} \int_o^{u_1/a} \frac{\exp(t)}{a} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+2)!\,r!} \left(\frac{Q_2}{b}\right)^{r+2} \left(\frac{u_1}{a} - t\right)^r dt du_1. \tag{30}$$

By adding (29) and (30) Δ is obtained as

$$\Delta = \exp\left(\frac{Q_1}{a}\right) - 1. \tag{31}$$

The expression for N_1 is given by (18). The single integral is

(32)

$$\int_{R_1} u_1\tau_1(u_1)du_1 = \int_o^{Q_1} \frac{u_1}{a}\left\{\left(1-\frac{Q_2}{b}\right)\exp\left(\frac{u_1}{a}\right)\right.$$

$$\left.+\frac{Q_2}{b}\int_o^{u_1/a}\frac{\exp(t)}{\frac{u_1}{a}-t}J_2\left[2\sqrt{\frac{Q_2}{b}\left(\frac{u_1}{a}-t\right)}\right]dt\right\}du_1$$

$$= a\left(1-\frac{Q_2}{b}\right)\left[\left(\frac{Q_1}{a}-1\right)\exp\left(\frac{b_1}{a}\right)+1\right]$$

$$+\int_o^{Q_1}\int_o^{u_1/a}\frac{u_1\exp(t)}{a}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+2)!r!}\left(\frac{Q_2}{b}\right)^{r+2}\left(\frac{u_1}{a}-t\right)^r dtdu_1.$$

The double integral is

$$\iint_R u_1\tau(\underline{u})d\underline{u} = \int_o^{Q_1}\int_o^{Q_2}\frac{u_1}{ab}\left\{\exp\left(\frac{u_1}{a}\right)-\sqrt{\frac{Q_2-u_2}{b}}\int_o^{u_1/a}\frac{\exp(t)}{\sqrt{\frac{u_1}{a}-t}}J_1\left[2\sqrt{\left(\frac{Q_2-u_2}{b}\right)\left(\frac{u_1}{a}-t\right)}\right]dt\right\}du_2du_1$$

$$= \frac{aQ_2}{b}\left[\left(\frac{Q_1}{a}-1\right)\exp\left(\frac{Q_1}{a}\right)+1\right]-\int_o^{Q_1}\int_o^{Q_2}\int_o^{u_1/a}\frac{u_1\exp(t)}{ab}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+1)!r!}\left(\frac{Q_2-u_2}{b}\right)^{r+1}\left(\frac{u_1}{a}-t\right)^r dtdu_2du_1.$$

Changing the order of integration for u_2 and t and showing the series converges uniformly for $u_2[0, Q_2]$, yields

(33)
$$\iint_R u_1\tau(\underline{u})d\underline{u} = \frac{aQ_2}{b}\left[\left(\frac{Q_1}{a}-1\right)\exp\left(\frac{Q_1}{a}\right)+1\right]$$

$$-\int_o^{Q_1}\int_o^{u_1/a}\frac{u_1\exp(t)}{a}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+2)!r!}\left(\frac{Q_2}{b}\right)^{r+2}\left(\frac{u_1}{a}-t\right)dtdu_1.$$

Adding (32) and (33) yields

(34)
$$N_1 = a\left[\left(\frac{Q_1}{a}-1\right)\exp\left(\frac{Q_1}{a}\right)+1\right].$$

The expression for N_2 is given by (20).

$$N_2 = \int_o^{Q_1} \int_o^{Q_2} \frac{u_2}{ab} \left\{ \exp\left(\frac{u_1}{a}\right) - \sqrt{\frac{Q_2 - u_2}{b}} \int_o^{u_1/a} \frac{\exp(t)}{\sqrt{\frac{u_1}{a} - t}} J_1\left[2\sqrt{\left(\frac{u_2 - u_2}{b}\right)\left(\frac{u_1}{a} - t\right)}\right] dt \right\} du_2 du_1$$

$$= \frac{Q_2^2}{2b}\left[\exp\left(\frac{Q_1}{a}\right) - 1\right] - \int_o^{Q_1} \int_o^{Q_2} \int_o^{u_1/a} \frac{u_2 \exp(t)}{ab} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+1)!\,r!} \left(\frac{Q_2 - u_2}{b}\right)^{r+1} \left(\frac{u_1}{a} - t\right)^r dt du_2 du_1.$$

Changing the order of integration for u_2 and t and showing the series converges uniformly for $u_2 \epsilon [0, Q_2]$ yields

$$N_2 = \frac{Q_2^2}{2b}\left[\exp\left(\frac{Q_1}{a}\right) - 1\right] - \int_o^{Q_1} \int_o^{u_1/a} \frac{b \exp(t)}{a} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+3)!\,r!} \left(\frac{Q_2}{b}\right)^{r+3} \left(\frac{u_1}{a} - t\right)^r dt du_1.$$

Changing the order of integration for u_1 and t and showing that the series converges uniformly for $u_1 \epsilon [at_1, Q_1]$ yields

$$N_2 = \frac{Q_2^2}{2b}\left[\exp\left(\frac{Q_1}{a}\right) - 1\right] - \int_o^{Q_1/a} b \exp(t) \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+3)!(r+1)!} \left(\frac{Q_2}{b}\right)^{r+3} \left(\frac{Q_1}{a} - t\right)^{r+1} dt.$$

Making the change in variable $x = \frac{Q_1}{a} - t$ and showing the series converges uniformly for $x \epsilon \left[0, \frac{Q_1}{a}\right]$ yields

$$N_2 = \frac{Q_2^2}{2b}\left[\exp\left(\frac{Q_1}{a}\right) - 1\right] - b \sum_{r=0}^{\infty} \frac{(-r)^r}{(r+3)!} \left(\frac{Q_2}{b}\right)^{r+3} \exp\left(\frac{Q_1}{a}\right) \int_o^{Q_1/a} \frac{\exp(-x) x^{r+1}}{(r+1)!} dx$$

$$= \frac{Q_2^2}{2b}\left[\exp\left(\frac{Q_1}{a}\right) - 1\right] - b \exp\left(\frac{Q_1}{a}\right) \sum_{r=0}^{\infty} \frac{(-r)^r}{(r+3)!} \left(\frac{Q_2}{b}\right)^{r+3} P\left(r+2, \frac{Q_1}{a}\right), \tag{35}$$

where $P\left(r+2, \frac{Q_1}{a}\right)$ is an Incomplete Gamma Function.

The expression for M_1 is given by (22). The single integral is

$$\int_{R_1} u_1^2 \tau_1(u_1) du_1 = \int_o^{Q_1} \frac{u_1^2}{a} \left\{ \left(1 - \frac{Q^2}{b}\right) \exp\left(\frac{u_1}{a}\right) + \frac{Q_2}{b} \int_o^{u_1/a} \frac{\exp(t)}{\frac{u_1}{a} - t} J_2\left[2\sqrt{\frac{Q_2}{b}\left(\frac{u_1}{a} - t\right)}\right] dt \right\} du_1$$

or

$$\int_{R_1} u_1^2 \tau_1(u_1) du_1 = a^2 \left(1 - \frac{Q_2}{b}\right) \left[\left(\frac{Q_1^2}{a^2} - 2\frac{Q_1}{a} + 2\right) \exp\left(\frac{Q_1}{a}\right) - 2\right] \tag{36}$$

$$+ \int_o^{Q_1} \int_o^{u_1/a} \frac{u_1^2 \exp(t)}{a} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+2)!\,r!} \left(\frac{Q_2}{b}\right)^{r+2} \left(\frac{u_1}{a} - t\right)^r dt du_1.$$

The double integral is

$$\int\int_R u_1^2\,\tau(\underline{u})d\underline{u} = \int_o^{Q_1}\int_o^{Q_2}\frac{u_1^2}{ab}\left\{\exp\left(\frac{u_1}{a}\right)\right.$$

$$\left.-\sqrt{\frac{Q_2-u_2}{b}}\int_o^{u_1/a}\frac{\exp(t)}{\sqrt{\frac{u_1}{a}-t}}J_1\left[2\sqrt{\left(\frac{Q_2-u_2}{b}\right)\left(\frac{u_1}{a}-t\right)}\right]dt\right\}du_2du_1$$

$$=\frac{a^2Q_2}{b}\left[\left(\frac{Q_1^2}{a^2}-2\frac{Q_1}{a}+2\right)\exp\left(\frac{Q_1}{a}\right)-2\right]$$

$$-\int_o^{Q_1}\int_o^{Q_2}\int_o^{u_1/a}\frac{u_1^2\exp(t)}{ab}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+1)!r!}\left(\frac{Q_2-u_2}{b}\right)^{r+1}\left(\frac{u_1}{a}-t\right)dtdu_2du_1.$$

Changing the order of integration for u_2 and t and showing the series converges uniformly for $u_2\epsilon[0, Q_2]$ yields

$$\int\int_R u_1^2\,\tau(\underline{u})d\underline{u} = \frac{a^2Q_2}{b}\left[\left(\frac{Q_1^2}{a^2}-2\frac{Q_1}{a}+2\right)\exp\left(\frac{Q_1}{a}\right)-2\right]$$

$$-\int_o^{Q_1}\int_o^{Q_2}\frac{u_1^2\exp(t)}{a}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+2)!r!}\left(\frac{Q_2}{b}\right)^{r+2}\left(\frac{u_1}{a}-t\right)^r dtdu_1. \tag{37}$$

Adding (36) and (37) yields

$$M_1 = a^2\left[\left(\frac{Q_1^2}{a^2}-2\frac{Q_1}{a}+2\right)\exp\left(\frac{Q_1}{a}\right)-2\right]. \tag{38}$$

The expression for M_2 is given by (24).

$$M_2 = \int_o^{Q_1}\int_o^{Q_2}\frac{u_2^2}{ab}\left\{\exp\left(\frac{u_1}{a}\right)-\sqrt{\frac{Q_2-u_2}{b}}\int_o^{u_1/a}\frac{\exp(t)}{\sqrt{\frac{u_1}{a}-t}}J_1\left[2\sqrt{\left(\frac{Q_2-u_2}{b}\left(\frac{u_1}{a}-t\right)\right)}\right]dt\right\}du_2du_1.$$

Changing the order of integration for u_2 and t and showing the series converges uniformly for $u_2\epsilon[0, Q_2]$ yields

$$M_2 = \frac{Q_2^3}{3b}\left[\exp\left(\frac{Q_1}{a}\right)-1\right]-\int_o^{Q_1}\int_o^{u_1/a}\frac{2b^2\exp(t)}{a}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+4)!r!}\left(\frac{Q_2}{b}\right)^{r+4}\left(\frac{u_1}{a}-t\right)^r dtdu_1.$$

It is seen from the calculation of N_2 that

$$M_2 = \frac{Q_2^3}{3b}\left[\exp\left(\frac{Q_1}{a}\right)-1\right]-2b^2\exp\left(\frac{Q_1}{a}\right)\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+4)!}\left(\frac{Q_2}{b}\right)^{r+4}P\left(r+2,\frac{Q_1}{a}\right). \tag{39}$$

From (29)

$$\int_{R_1}\tau_1(u_1)du_1=\left(1-\frac{Q_2}{b}\right)\left[\exp\left(\frac{Q_1}{a}\right)-1\right]$$

$$+\int_o^{Q_1}\int_o^{\frac{u_1}{a}}\frac{\exp(t)}{a}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+2)!r!}\left(\frac{Q_2}{b}\right)^{r+2}\left(\frac{u_1}{a}-t\right)^r dtdu_1 .$$

From the calculation of N_2 it is seen that

$$\int_{R_1}\tau_1(u_1)du_1=\left(1-\frac{Q_2}{b}\right)\left[\exp\left(\frac{Q_1}{a}\right)-1\right]+\exp\left(\frac{Q_1}{a}\right)\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+2)!}\left(\frac{Q_2}{b}\right)^{r+2}P\left(r+2,\frac{Q_1}{a}\right). \tag{40}$$

Substituting (31), (34), (35), (38), (39), and (40) into (28) yields

$$\bar{E}=\tfrac{1}{2}(p_1a+p_2b)+K\exp\left(-\frac{Q_1}{a}\right)+K_2\left\{\left(1-\frac{Q_2}{b}\right)\left[1-\exp\left(-\frac{Q_1}{a}\right)\right]\right. \tag{41}$$

$$\left.+\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+2)!}\left(\frac{Q_2}{b}\right)^{r+2}P\left(r+2,\frac{Q_1}{a}\right)\right\}+p_1a\left[\left(\frac{Q_1}{a}-1\right)+\exp\left(-\frac{Q_1}{a}\right)\right]$$

$$+p_2b\left\{\tfrac{1}{2}\left(\frac{Q_2}{b}\right)^2\left[1-\exp\left(-\frac{Q_1}{a}\right)\right]-\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+3)!}\left(\frac{Q_2}{b}\right)^{r+3}P\left(r+2,\frac{Q_1}{a}\right)\right\}$$

$$+\tfrac{1}{2}a(h_1+p_1)\left[\frac{Q_1^2}{a^2}-2\frac{Q_1}{a}+2-2\exp\left(-\frac{Q_1}{a}\right)\right]$$

$$+\tfrac{1}{2}b(h_2+p_2)\left\{\tfrac{1}{3}\left(\frac{Q_2}{b}\right)^3\left[1-\exp\left(-\frac{Q_1}{a}\right)\right]-2\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+4)!}\left(\frac{Q_2}{b}\right)^{r+4}P\left(r+2,\frac{Q_1}{a}\right)\right\}$$

$$-\tfrac{1}{2}a(h_1+p_1)\left\{\frac{p_1}{h_1+p_1}+\left[\frac{Q_1}{a}-1+\exp\left(-\frac{Q_1}{a}\right)\right]\right\}^2$$

$$-\tfrac{1}{2}b(h_2+p_2)\left\{\frac{p_2}{h_2+p_2}+\tfrac{1}{2}\left(\frac{Q_2}{b}\right)^2\left[\left(1-\exp\left(\frac{Q_1}{a}\right)\right]-\sum_{r=0}^{\infty}\frac{(-1)^r}{(r+3)!}\left(\frac{Q_2}{b}\right)^{r+3}P\left(r+2,\frac{Q_1}{a}\right)\right\}^2.$$

From (26) and (27)

$$S_1^* = \frac{p_1 a}{h_1 + p_1} + \frac{N_1}{1+\Delta}\Big|_{Q_1^*, Q_2^*}$$

$$S_2^* = \frac{p_2 b}{h_2 + p_2} + \frac{N_2}{1+\Delta}\Big|_{Q_1^*, Q_2^*}.$$

Example

A computer program that determines Q_1^* and Q_2^* has been written using a course grid search method. Since (41) is a function of exponential terms, steep, narrow peaks will not exist. While it has not been shown, all surfaces of (41) that have been determined have been unimodal.

Let

$$\begin{aligned} c_1 &= 2, & c_2 &= 2 \\ p_1 &= 30, & p_2 &= 16 \\ h_1 &= 2, & h_2 &= 3 \\ a &= 3, & b &= 9 \\ K &= 20. \end{aligned}$$

To determine regions in which either the mixed ordering policy or the joint individual ordering policy [9] or the joint ordering policy has the least expected cost as K_1 and K_2 vary according to $\max(K_1, K_2) \leq K \leq K_1 + K_2$, the following values of K_1 and K_2 will be used

		K_2				
		0	5	10	15	20
	0					X
	5				X	X
K_1	10			X	X	X
	15		X	X	X	X
	20	X	X	X	X	X.

The equation for $\bar{E}$ is given by (41). The subscript $i (i = 1, 2)$ will be used to indicate which product can be ordered individually. The expected cost for the joint individual ordering policy is given by

$$\bar{E}_I = \sum_1^2 \left\{ p_i a_i \cdot \tfrac{1}{2}\left[1 + \left(\frac{Q_i}{a_i}\right)^2\right] + K_i \exp\left(-\frac{Q_i}{a_i}\right) + h_i a_i \tfrac{1}{2}\left[\left(\frac{Q_i}{a_i}\right)^2 - 2\frac{Q_i}{a_i} + 2 - 2\exp\left(-\frac{Q_i}{a_i}\right)\right] \right.$$
$$\left. -\tfrac{1}{2}(h_i + p_i) a_i \left[\frac{p_i}{h_i + p_i} + \frac{Q_i}{a_i} + \exp\left(-\frac{Q_i}{a_i}\right)\right]^2 \right\} + (K - K_1 - K_2) \exp\left(\frac{Q_1}{a_1}\right) \exp\left(-\frac{Q_2}{a_2}\right),$$

where

$$a_i = \begin{cases} a, & i = 1 \\ b, & i = 2. \end{cases}$$

The expected cost for the joint ordering policy is given by

$$\bar{E}_J=\frac{K}{1+\Delta}+\sum_1^2\left\{\tfrac{1}{2}p_ia_i+\frac{1}{1+\Delta}\left[p_iN_i+\tfrac{1}{2}(h_i+p_i)\frac{M_i}{a_i}\right]-\frac{h_i+p_i}{2a_i}\left(\frac{p_ia_i}{h_i+p_i}+\frac{N_i}{1+\Delta}\right)^2\right\},$$

where

$$\Delta=I_0\left[2\sqrt{\frac{Q_1Q_2}{ab}}\right]-1$$

$$N_i=Q_iI_2\left[2\sqrt{\frac{Q_1Q_2}{ab}}\right]$$

$$M_i=Q_i^2\left\{I_4\left[2\sqrt{\frac{Q_1Q_2}{ab}}\right]+2\sqrt{\frac{ab}{Q_1Q_2}}I_3\left[2\sqrt{\frac{Q_1Q_2}{ab}}\right]\right\},$$

and

$I_n(\cdot)$ is a modified Bessel function of order n.

The equation for $\overline{E_J}$ is independent of K_1 and K_2 and thus as K_1 and K_2 vary, $\overline{E_J}$ will remain constant. Thus, for all K_1 and K_2

$$\overline{E_J}=31.93,\qquad S_1^*=3.00,\qquad S_2^*=8.03,\qquad Q_1^*=1.68,\qquad Q_2^*=4.14.$$

The equation for $\overline{E_1}$ is independent of K_2 and thus

$$\overline{E_1}, S_1^*, S_2^*, Q_1^*, Q_2^*=\begin{cases}28.49, 2.81, 8.61, 0.00, 4.68: K_1=0\\ 30.02, 2.87, 8.40, 1.02, 4.14: K_1=5\\ 31.10, 2.93, 8.27, 1.44, 3.78: K_1=10\\ 31.90, 2.98, 8.15, 1.80, 3.42: K_1=15\\ 32.49, 3.00, 7.99, 2.04, 2.88: K_1=20.\end{cases}$$

The equation for $\overline{E_2}$ is independent of K_1 and thus

$$\overline{E_2}, S_1^*, S_2^*, Q_1^*, Q_2^*=\begin{cases}28.51, 2.98, 7.58, 1.08, 0.00: K_2=0\\ 29.84, 2.98, 7.65, 1.08, 2.16: K_2=5\\ 30.94, 2.98, 7.75, 1.08, 3.24: K_2=10\\ 31.89, 2.98, 7.83, 1.08, 3.96: K_2=15\\ 32.75, 2.98, 7.91, 1.08, 4.50: K_2=20.\end{cases}$$

The following matrix displays the values of $\overline{E_I}$ as K_1 and K_2 vary:

		K_2				
		0	5	10	15	20
	0					28.49
	5				29.43	30.19
K_1	10			29.56	30.63	31.58
	15		29.33	30.52	31.68	32.74
	20	28.51	29.95	31.32	32.54	33.70.

The following matrix displays S_1^*, S_2^*, Q_1^*, and Q_2^* for the joint individual ordering policy as K_1 and K_2 vary in the format $\begin{matrix} S_1^*, S_2^* \\ Q_1^*, Q_2^* \end{matrix}$

		K_2				
		0	5	10	15	20
	0					2.81, 8.61 0.00, 4.68
	5				2.97, 8.40 1.02, 4.14	2.86, 8.40 0.60, 4.14
K_1	10			2.98, 8.10 1.08, 3.24	2.98, 8.21 1.08, 3.60	2.92, 8.21 0.84, 3.60
	15		2.98, 7.86 1.08, 2.34	2.98, 7.90 1.08, 2.52	2.98, 7.99 1.08, 2.88	2.95, 8.04 0.96, 3.06
	20	2.98, 7.58 1.08, 0.00	2.98, 7.66 1.08, 1.26	2.98, 7.75 1.08, 1.80	2.98, 7.82 1.08, 2.16	2.95, 7.86 0.96, 2.34

By comparing the preceding expected costs, a matrix can be developed whose elements (M_1, M_2, I, J) indicate which ordering policy has the least expected cost for the designated K_1 and K_2. M_i ($i = 1, 2$) is the simplest mixed ordering policy which allows product i to be ordered individually. I and J designate the joint individual ordering policy and the joint ordering policy, respectively.

		K_2				
		0	5	10	15	20
	0					M_1, I
	5				I	M_1
K_1	10			I	I	M_1
	15		I	I	I	M_1
	20	M_2, I	M_2	M_2	M_2	J

If the restrictions that $s_1' = s_1''$ and $s_2' = S_2$ are removed in the development of the optimal ordering rules for the mixed ordering policy, then there will be fewer regions in which the joint individual ordering policy and the joint ordering policy will have lesser expected cost than the mixed ordering policy.

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STOCHASTIC CONTROL OF QUEUEING SYSTEMS*

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ABSTRACT

Suppose that the state of a queueing system is described by a Markov process $\{Y_t, t \geq 0\}$, and the profit from operating it up to a time t is given by the function $f(Y_t)$. We operate the system up to a time T, where the random variable T is a stopping time for the process Y_t. Optimal stochastic control is achieved by choosing the stopping time T that maximizes $Ef(Y_T)$ over a given class of stopping times. In this paper a theory of stochastic control is developed for a single server queue with Poisson arrivals and general service times.

1. INTRODUCTION

A class of control problems in a queueing system is concerned with how best to operate it over time, given the cost structure. Specifically, suppose that the state of the system is described by a Markov process $\{Y_t, t \geq 0\}$, and the profit from operating the system up to a time t is given by a function $f(Y_t)$. Deterministic control of the system is achieved by operating the system up to a time $\hat{t}$, where $\hat{t}$ is such that $Ef(Y_t)$ is a maximum at $\hat{t}$. In stochastic control theory we operate the system up to a random time T, whose observed value (say t) will depend only on the history of the process up to time t. Such a random variable T is called a stopping time for the process Y_t. Optimal stochastic control is achieved by choosing the stopping time T that maximizes $Ef(Y_T)$ over a given class of stopping times.

The theory of optimal stopping times for Markov processes was developed by Dynkin [1]. Further discussion and several examples are given by Dynkin and Yushkevitch [2]. Modifications of Dynkin's criteria are investigated by Taylor [7] and Ross [6].

Queues are notorious for giving rise to non-Markovian processes, so it is not possible to develop a general theory of stochastic control of queueing systems. Accordingly, we consider a system subject to the following assumptions:

(1) Customers arrive in a Poisson process at a rate λ $(0 < \lambda < \infty)$.

(2) The service times of successive customers are random variables which are independent, and also independent of the arrival process, and have the common distribution function $(d.f.)$ $B(x)$.

(3) There is one server, and the queue-discipline is "first-come, first-served."

Let W_t be the remaining workload of the server at time t; $W_t = 0$ if the system is empty at time t, and otherwise W_t is the sum of the residual service time of the customer being served and the service times of waiting customers. [In the queueing literature W_t is also called the virtual waiting time.] It is known (see for example, Prabhu [5]) that $\{W_t, t \geq 0\}$ is a time-homogeneous Markov process with the state-space $[0, \infty)$. Our next assumption is the following:

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(4) The profit from operating the system up to time t is given by $f(t, W_t)$. Here $f(t, x)$ is a bounded continuous function of $t \geq 0$, $x \geq 0$.

It is clear that $Y_t = (t, W_t)$ is also a time-homogeneous Markov process with the state-space $[0, \infty) \times [0, \infty)$. The profit function can be denoted as $f(Y_t)$. Let $Y_0 = y$ with probability one. Our objective is to find the stopping time T_0 for which

$$Ef(Y_{T_0}) \geq Ef(Y_T), \tag{1}$$

for all stopping times T in a given class; we shall call T_0 optimal at y.

The theory of optimal stopping time for a Markov process uses the concept of the infinitesimal generator of the associated semigroup of operators. In section 2 this semigroup is defined and its generator obtained. The optimal stopping time for a class of profit functions is given by a result due to Ross [6]. This result is then applied to a special case in section 3. Finally in section 4 some further remarks are made on stochastic control of queues.

2. THE INFINITESIMAL GENERATOR OF Y_t

The remaining workload process W_t described in the last section is a Markov process of the mixed type, whose sample function increases only by jumps, and decreases continuously at a unit rate except if $W_t = 0$, when it continues to be zero until the next jump occurs (see Figure 1). Moreover,

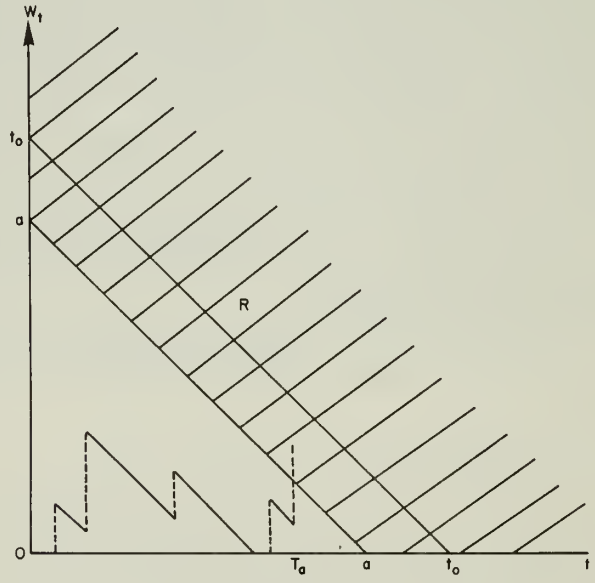

FIGURE 1.

the successive jumps are independent random variables with the common *d.f.* $B(x)$. Let us denote its transition *d.f.* by

$$F(x; y, t) = P\{W_{s+t} \leq y \mid W_s = x\}. \tag{2}$$

An explicit expression for $F(x; y, t)$ is known (see Prabhu [5]). In particular we have for $h \to 0+$,

$$F(x; y, h) = (1-\lambda h)B_0(y-x+h) + \lambda h B(y-x+h) + o(h), \tag{3}$$

where $B_0(\theta) = 0$ for $\theta < 0$, and $= 1$ for $\theta \geq 0$.

The process $Y_t = (t, W_t)$ is also a time-homogeneous Markov process. Its transition $d.f.$ is given by

$$\begin{aligned} H(s, x; s', y, t) &= P\{\tau + t \leq s', W_{\tau+t} \leq y | \tau = s, W_\tau = x\} \\ &= P\{s + t \leq s', W_{s+t} \leq y | W_s = x\} \\ &= F(x; y, t) \qquad \text{if } s + t \leq s' \\ &= 0 \qquad\qquad \text{if } s + t > s'. \end{aligned}$$

We can write this as

$$H(s, x; s', y, t) = F(x; y, t)B_0(s'-s-t). \tag{4}$$

Associated with the process Y_t we have the semigroup $\{T_t, t > 0\}$ of operators defined by the following:

$$T_t f(s, x) = \int_0^\infty \int_0^\infty f(s', y) H(s, x; ds', dy, t).$$

We take the domain of T_t to be the class C of bounded continuous functions on $[0, \infty) \times [0, \infty)$. Using (4) we can simplify the last equation as

$$T_t f(s, x) = \int_{0-}^{\infty} f(s + t, y) F(x; dy, t). \tag{5}$$

Let $T_0 = I$, the identity operator. The semigroup $\{T_t\}$ is continuous since

$$\begin{aligned} T_h f(t, x) - f(t, x) &= \int_{0-}^{\infty} f(t + h, y) F(x; dy, h) - f(t, x) \\ &= \int_{0-}^{\infty} f(t + h, y)[(1 - \lambda h)B_0(dy - x + h) + \lambda h B(dy - x + h) + o(h)] - f(t, x) \\ &= f(t + h, x - h) - f(t, x) + o(1) \to 0 \end{aligned}$$

as $h \to 0+$.

The infinitesimal generator of $\{T_t\}$ is the linear operator A defined by

$$Af(t, x) = \lim_{h \to 0+} \frac{T_h f(t, x) - f(t, x)}{h}, \tag{6}$$

for those functions $f(t, x)$ in C for which this limit exists. Theorem 1 below gives this generator.

THEOREM 1: Let the function $f(t, x)$ and its partial derivatives $\partial f/\partial t$, $\partial f/\partial x$ be bounded and continuous. Then the infinitesimal generator of the semigroup $\{T_t\}$ is given by

$$Af(t, x) = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} - \lambda f(t, x) + \lambda \int_{0-}^{\infty} f(t, x+v) B(dv). \tag{7}$$

PROOF: We have

$$\begin{aligned}
h^{-1}[T_h f(t, x) - f(t, x)] &= h^{-1} \int_{0-}^{\infty} [f(t+h, y) - f(t, x)] F(x; dy, h) \\
&= h^{-1}[f(t+h, x-h) - f(t, x)](1 - \lambda h) \\
&\quad + \lambda \int_{x-h-}^{\infty} [f(t+h, y) - f(t, x)] B(dy - x + h) + h^{-1} o(h) \\
&= h^{-1}[f(t+h, x-h) - f(t, x-h)] + h^{-1}[f(t, x-h) - f(t, x)] \\
&\quad - \lambda f(t+h, x-h) + \lambda \int_{0-}^{\infty} f(t+h, v+x-h) B(dv) + h^{-1} o(h) \\
&\to \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} - \lambda f(t, x) + \lambda \int_{0-}^{\infty} f(t, x+v) B(dv)
\end{aligned}$$

as $h \to 0+$, as required.

The optimal stopping time that we are seeking is based on the infinitesimal look-ahead rule which can be described roughly as follows. Suppose that the system has been in operation up to a time t, and a profit $f(t, x)$ has been made. Comparing $f(t, x)$ with $f(t+h, W_{t+h})$ we see that it will be profitable to continue the operation up to time $t+h$ if and only if

$$Ef(t+h, W_{t+h}) > f(t, x).$$

Proceeding as in the proof of Theorem 1, we can simplify this last inequality as

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} - \lambda f(t, x) + \lambda \int_{0-}^{\infty} f(t, x+v) B(dv) + h^{-1} o(h) > 0. \tag{8}$$

Let us denote the left side of (8) as $g(t, x, h)$. Then the above argument shows that the operation of the system is to be continued until a time t' such that $g(t', x, h) \leq 0$. This intuitive approach does not completely solve the problem since it is not clear that we will have $g(t, x, h) \leq 0$ for all $t \geq t'$. We need to ensure this, and also that with probability one the operation of the system will terminate eventually. In the following theorem conditions (11) and (12) ensure that these conditions will be met. It should be

noted that as $h \to 0+$, $g(t, x, h) \to Af(t, x)$. We first introduce the following notation.

For any subset A of the state-space of Y_t we denote

$$T_A = \inf \{t > 0 : Y_t \in A\}, \tag{9}$$

so that T_A is the time of the first visit to A. As usual we define $T_A = +\infty$ is the set of t-points in (9) is empty. The following result is due to Ross [6], but his proof is sketchy.

THEOREM 2: Let R be the set

$$R = \{(t, x) : Af(t, x) \leq 0\}, \tag{10}$$

and $\bar{R}$ the complement of R. If the set R is closed in the sense that

$$P\{T_{\bar{R}} < \infty | Y_0 \in R\} = 0 \tag{11}$$

and if

$$P\{T_R < \infty | Y_0 \in \bar{R}\} = 1, \tag{12}$$

then T_R is the optimal stopping time for $Y_0 \in \bar{R}$.

PROOF: We have Dynkin's [1] result

$$Ef(T, W_T) - f(0, W_0) = E \int_0^T Af(s, W_s) ds. \tag{13}$$

which holds for all stopping times T with $E(T) < \infty$. For any such stopping time T we therefore have

$$Ef(T_R, W_{T_R}) - Ef(T, W_T) = E \int_0^{T_R} Af(s, W_s) ds - E \int_0^T Af(s, W_s) ds$$

$$= E\chi_{T<T_R} \int_T^{T_R} Af(s, W_s) ds - E(1 - \chi_{T<T_R}) \int_{T_R}^T Af(s, W_s) ds,$$

where χ_E is the indicator function of the event E, that is, $\chi_E = 1$ or 0 according as E occurs or does not occur. Since $Af(s, W_s) > 0$ for $s < T_R$ and ≤ 0 for $s > T_R$ it follows that

$$Ef(T_R, W_{T_R}) - Ef(T, W_T) > 0.$$

This establishes the optimality of T_R. [It has been tacitly assumed that $E(T_R) < \infty$.]

3. AN EXAMPLE

To illustrate the concept of stochastic control of queues we consider the system described in section 1. Suppose it is decided to admit and serve all customers arriving up to time T, and reject all subsequent arrivals. Let the revenue and cost factors be given as follows:

r = revenue per unit time

c_1 = operating cost per unit time in $(0, t_0]$

c_2 = increased operating cost per unit time in (t_0, ∞).

Here t_0 is an epoch of time such that overtime wages will have to be paid if the servicing of customers continues beyond t_0. We assume that $c_1 < r < c_2$, and put $d_1 = r - c_1$, $d_2 = c_2 - r$. The profit from such an operation of the system up to a time T is given by $f(T, W_T)$, where

$$f(t, x) = \int_0^{t+x} e^{-\alpha s} p(s)\, ds. \tag{14}$$

Here $e^{-\alpha t}$ $(\alpha > 0)$ is a discount factor, and

$$\begin{aligned} p(t) &= d_1 \qquad \text{for } t \leqslant t_0 \\ &= -d_2 \qquad \text{for } t > t_0. \end{aligned} \tag{15}$$

A profit function similar to the one above was considered by Mine and Ohno [4], who obtained the solution to the deterministic control problem. Here we are interested in stochastic control. It should be noted, however, that T is a rejection time rather than a stopping time. See also Emmons [3].

THEOREM 3: Let $W_0 = 0$, and assume that

$$\int_0^{t_0} e^{-\alpha v}[1 - B(v)\, dv > \frac{d_2}{d_1 + d_2} \cdot \frac{1 - \psi(\alpha)}{\alpha}, \tag{16}$$

where

$$\psi(\alpha) = \int_0^{\infty} e^{-\alpha v} B(dv). \tag{17}$$

Then the optimal time is given by

$$T_a = \inf\{t > 0 : t + W_t \geqslant a\}, \tag{18}$$

where a is the unique solution of the equation

$$(d_1 + d_2) \int_0^{t_0 - t} e^{-\alpha v}[1 - B(v)]\, dv - d_2. \quad \frac{1 - \psi(\alpha)}{\alpha} = 0. \tag{19}$$

PROOF: Proceeding as in the proof of Theorem 1 we see that $Af(t, x) = \lambda \psi(t + x)$, where

$$\psi(t) = \int_0^{\infty} e^{-\alpha(t+v)} p(t + v)[1 - B(v)]\, dv. \tag{20}$$

In view of (15) we find that

$$\psi(t) = (d_1+d_2)e^{-\alpha t}\int_0^{t_0-t} e^{-\alpha v}[1-B(v)]dv - d_2e^{-\alpha t}\cdot\frac{1-\psi(\alpha)}{\alpha} \quad (t \leqslant t_0) \tag{21}$$

$$= -d_2e^{-\alpha t}\cdot\frac{1-\psi(\alpha)}{\alpha} \quad (t \geqslant t_0).$$

In the interval $[0, t_0]$ we have

$$\psi'(t) = (d_1+d_2)(-\alpha)e^{-\alpha t}\int_0^{t_0-t} e^{-\alpha v}[1-B(v)]dv$$

$$-(d_1+d_2)e^{-\alpha t}e^{-\alpha(t_0-t)}[1-B(t_0-t)] + d_2e^{-\alpha t}[1-\psi(\alpha)]$$

$$= (d_1+d_2)e^{-\alpha t}\int_0^{t_0-t} e^{-\alpha v}B(dv) - d_1e^{-\alpha t} - d_2e^{-\alpha t}\psi(\alpha).$$

Since the integral in the last expression cannot exceed $\psi(\alpha)$ it follows that

$$\psi'(t) < -d_1e^{-\alpha t}[1-\psi(\alpha)] < 0 \qquad (t \leqslant t_0). \tag{22}$$

Therefore $\psi(t)$ is a decreasing function in $[0, t_0]$. We have

$$\psi(0) = (d_1+d_2)\int_0^{t_0} e^{-\alpha v}[1-B(v)]dv - d_2\cdot\frac{1-\psi(\alpha)}{\alpha},$$

and this last expression is positive by (18). Also

$$\psi(t_0) = -d_2e^{-\alpha t_0}\cdot\frac{1-\psi(\alpha)}{\alpha} < 0.$$

Therefore there exists a unique point a such that $\psi(t) > 0$ for $0 \leqslant t < a$ and $\psi(t) < 0$ for $a \leqslant t \leqslant t_0$. This point a is given by (19). Since $\psi(t) \leqslant 0$ for $t \geqslant t_0$, we have $\psi(t) \leqslant 0$ for $t \geqslant a$. This means that $Af(t, x) \leqslant 0$ for $t+x \geqslant a$, so the set R of Theorem 2 is given by $R = \{(t, x) : t+x \geqslant a\}$, and the time of the first visit to R is given by (18). Since the process $t+W_t$ is monotone nondecreasing with probability one, the set R is closed. Moreover, $T_a \leqslant a$ with probability one and also $E(T_a) < \infty$. Thus the conditions of Theorem 2 are satisfied, and T_a is indeed optimal at $W_0 = 0$, as was required to be proved.

A particular case: Let us consider the case where $B(x) = 1-e^{-\mu x}$ $(x \geqslant 0,\ 0 < \mu < \infty)$. The condition (16) reduces to

$$t_0 > (\mu+\alpha)^{-1}\log\left(1+\frac{d_2}{d_1}\right). \tag{23}$$

Equation (19) gives

$$a = t_0 - (\mu + \alpha)^{-1} \log\left(1 + \frac{d_2}{d_1}\right). \tag{24}$$

On account of (23), we have $a > 0$.

Figure 1 illustrates the way in which stochastic control will be achieved if the profit function is given by (14)–(15) and the condition (16) is satisfied.

4. SOME COMMENTS

The profit function considered here is $f(Y_t)$. Other possible functions are of the form

$$\int_0^t h(Y_s)ds \qquad \text{or } g(Y_t) - \int_0^t c(Y_s)ds. \tag{25}$$

In view of Dynkin's result (13) these lead to the same type of result for the optimal stopping time.

We have taken the basic process to be W_t, the remaining workload in the system considered. To compute W_t one needs to know the service times of arriving customers. This can be done if, in particular, these service times are all equal to a known constant (deterministic service). In the general case one might have some good estimates of the service times and the stochastic control scheme can be used.

In queues with Poisson arrivals and exponential service times the queue-length, Q_t, is known to be a Markov process, and the profit function can be expressed in terms of this process. The theory of stochastic control can be developed for this situation along the lines indicated in this paper.

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MAINTENANCE POLICIES WHEN FAILURE DISTRIBUTION OF EQUIPMENT IS ONLY PARTIALLY KNOWN

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ABSTRACT

An optimal schedule for checking an equipment subject to failure which can be detected by inspection only, is derived. Increasing failure rate and one percentile specify the otherwise unknown life distribution. Dynamic programming methodology yields the solution which minimizes the maximum expected cost. Numerical examples are presented and compared with models employing differing amounts of knowledge.

I. INTRODUCTION

Maintenance policies for stochastically failing equipment have been calculated mainly for the case where the failure distribution is assumed to be known, even though in real life situations the failure characteristics of the equipment are mostly not available, so that maintenance activities must start with a limited amount of knowledge.

Minimax policies were devised by [2] and [7] assuming lack of all knowledge, or, respectively, knowledge of a percentile of the failure distribution; in both cases a time, T, denoting the maximum duration of equipment's service, is given. For expected value of the failure distribution known, and an increasing failure rate (*IFR*), [6] and [3], give some bounds for costs and policies, which are based on [1]. If the family of the *f.d.* is given, but one or more parameters are unknown, adaptive policies which have a Bayes structure and are improved sequentially, can be devised (see [6] and [3]).

The life time distribution of an equipment is known to be F_o. Given certain costs an optimal maintenance policy P_o can be calculated which leads to the minimum loss L_o:

$$L_o = L(P_o, F_o) = \min_P L(P, F_o). \tag{1}$$

The methods devised for partial knowledge (as detailed above) stipulate a distribution F^{**} while subsequently deriving an optimal policy P^{**} with minimal loss L^{**}:

$$L^{**} = L(P^{**}, F^{**}) = \min_P L(P, F^{**}). \tag{2}$$

In reality, policy P^{**} is applied to distribution F_o, which is the actual distribution of the equipment, leading in consequence to a suboptimized L^*:

$$L^*=L(P^{**}, F_o). \tag{3}$$

In general, L^* can be larger or smaller than L^{**}, but the virtue of a minimax policy is that L^{**} represents generally an upper bound.

For a minimax policy we have

$$L^{**}=L(P^{**}, F^{**})=\min_P L(P, F^{**})=\min_P \max_F L(P, F), \tag{4}$$

and if a saddle point exists

$$L^{**}=\max_F L(P^{**}, F) \tag{5}$$

so that from (3) follows

$$L^{**}=\max_F L(P^{**}, F) \geq L^*=L(P^{**}, F_o). \tag{6}$$

Seeking an improved upper bound for the cost, we assume that the failure distribution $F(t)$ is of increasing (or constant) failure rate (*IFR*). Furthermore, as in [7], a certain percentile,

$$t_p=x: \qquad F(t=x)=p, \tag{7}$$

is given, and we know T, the maximum length of time the equipment is to be kept in service. Assumptions of this type can reasonably be made after relatively short experience with a certain equipment, and especially since *IFR* is a very widespread characteristic.

Numerical examples at the end of this study show that bounds on the loss, derived by our model, are better than those derived under the assumptions of [2] and [7].

II. THE MODEL

We shall apply our method to a preparedness policy model, which has been solved already by [1] extended and tabulated by [4] and [5], so that numerical comparison with an exact policy is relatively easy.

Equipments life time is given by random variable, t. Its density function is $f(t)$; $0 \leq t \leq \infty$, distribution function $F(t)$ and reliability function

$$F^*(t)=1-F(t).$$

Failure rate is defined according to [1] for any random variable u by:

$$r(u)=\frac{F(u+\Delta)-F(u)}{F^*(u)}; \qquad u \geq 0, \qquad \Delta > 0. \tag{8}$$

Increasing failure rate, *IFR*, signifies

$$r(u_1) \geq r(u_2), \qquad u_1 > u_2. \tag{9}$$

In our model we assume that upon detection of failure or at time, T, the problem ends (alternatively we can assume that a new problem of exactly the same nature starts at this point). This time period will be called a "cycle." We assume:

(a) A system failure is detected only through checking, which costs I per check. A check is carried out also at time T.

(b) The time elapsed between system failure and its discovery at the next check costs a per unit of time.

(c) Checking takes negligible time, the equipment cannot fail during a check and is not degraded by the check.

(d) A cost R is paid if the equipment fails before time T.

From [4] and [5] we have the expected loss L per cycle

$$L = I \sum_{i=0}^{n-1} F^*(t_i) + a \left[\sum_{i=0}^{n-1} F^*(t_i)(t_{i+1} - t_i) - \int_0^{t_n} F^*(t)\,dt \right] + RF(T) \tag{10}$$

where the checking policy P is the sequence:

$$P = \{t_i: \quad i = 0, 1, 2, \ldots, n; \quad t_0 < t_1 < \ldots < t_n, \quad t_0 = 0, \quad t_n = T\} \tag{11}$$

We wish to develop an optimal checking policy P^{**} according to (4).

A general relation showing dependence of L on the various parameters of the model can be given. Let $\mathbb{W}$ be the vector of influential parameters. $\mathbb{W}_n$ will be a normalized parameter vector

$$\mathbb{W}_n = \{I = 1, a, R, T = 1, x, p\} \tag{12}$$

and $\mathbb{W}_u$ a nonnormalized vector

$$\mathbb{W}_u = \{I, a', R', T, x', p\}, \tag{13}$$

based on (10) we can then write directly (see also [4]) the following relations which exist simultaneously:

$$t_i(\mathbb{W}_u) = T t_i(\mathbb{W}_n), \quad i = 1, 2, \ldots, n \tag{14}$$

$$L(\mathbb{W}_u) = I L(\mathbb{W}_n)$$

if

$$a' = \frac{aI}{T}, \qquad R' = RI, \qquad x' = xT.$$

With the help of (12) to (14) we can derive solutions for a nonnormalized case if solutions for the nor-

malized parameters are available. Thus examples at the end of this study are given for normalized $W_n=\{I=1, a, R=0, T=2, x, p\}$.

III. MATHEMATICAL PREPARATIONS

According to [1], *a f.d.* $F(u)$, $u>0$ which is *IFR*, crosses the exponential distribution $1-e^{-\alpha u}$, $\alpha>0$ from below at most once in addition to coincidence at $u=0$ and ∞ unless they coincide identically. Consequently for $F(u=x)=p$ the common point, we obtain:

$$F(u)\leqslant 1-e^{-\alpha u} \qquad \text{if } u\leqslant x \tag{15}$$

$$F(u)\geqslant 1-e^{-\alpha u} \qquad \text{if } u\geqslant x$$

$$\alpha=-\frac{\ln(1-p)}{x}.$$

We now introduce a transformed exponential distribution function — *t.e.d.:*

$$F(v)=\begin{cases}0 & \text{if } v<d\\ 1-e^{-c(v-d)} & \text{if } v\geqslant d\\ & c>0,\end{cases} \tag{16}$$

from (8) it follows that the transformed exponential distribution is of constant failure rate

$$r(v)=1-e^{-c\Delta}, \qquad v>a, \qquad \Delta>0. \tag{17}$$

LEMMA 1: Generalizing properties of the exponential distribution (*e.d.*), the *t.e.d.* can be crossed by an *IFR* distribution at most twice, the first taking place from above.

PROOF: Assume that $F(t)$, which is *IFR*, crosses the *t.e.d.* $F(v)$, with constant failure rate $r(v)$, first from above and then from below. At the first coincidence $r(t_1)<r(v)$ and at the second $r(t_2)>r(v)$ so that $F(t)$ can indeed be *IFR*. A further crossing from above would mean $r(t_3)<r(v)$, contradicting the assumption of *IFR*.

Furthermore, we can uniquely find parameters c, d such that the *t.e.d.* passes through given two points (t_i, p_i): $p_i=F(t=t_i)$, $i=1, 2$.

LEMMA 2: Given three points

$$(t_i, p_i): \qquad p_i=F(w=t_i), \qquad i=0, 1, 2, \qquad t_0<t_1<t_2.$$

(a) An *IFR* distribution $F(w)$ passes through the three points only if (t_1, p_1) lies on or below the *t.e.d.* through (t_0, p_0), (t_2, p_2).

(b) The *IFR* distribution $F(w)$ which passes through the three points and possesses maximum area $\int_{t_0}^{t_2} F(w)dw$, is given by the two *t.e.d.* which meet in (t_1, p_1):

$$F_i(w)=1-e^{-c_i(w-d_i)} \tag{18}$$

$$c_i = \frac{1}{t_i - t_{i-1}} \ln\left(\frac{1-p_{i-1}}{1-p_i}\right)$$

$$d_i = \frac{1}{c_i} \ln(1-p_i) + t_i$$

$$t_{i-1} \leqslant w \leqslant t_i, \qquad i = 1, 2,$$

while

$$c_2 \geqslant c_1.$$

PROOF:

(a) This property is a direct result of Lemma 1.

(b) Define the following distribution functions:

t.e.d. connecting points (t_2, p_2), (t_0, p_0)

(19) $$F_3(w) = 1 - e^{-c_3(w-d_3)};$$

t.e.d. connecting points (t_1, p_1), (t_0, p_0)

(20) $$F_1(w) = 1 - e^{-c_1(w-d_1)};$$

t.e.d. connecting points (t_2, p_2), (t_1, p_1)

(21) $$F_2(w) = 1 - e^{-c_2(w-d_2)}.$$

We assume according to (a) that

(22) $$p_1 \leqslant 1 - e^{-c_3(t_1-d_3)}.$$

Since (20) crosses *t.e.d.* (19) from above it follows directly from definition (8) that:

(23) $$r_3(t_0) \geqslant r_1(t_0),$$

where $r_i(\cdot)$ is the failure rate for $F_i(\cdot)$. Similarly since (21) crosses (19) from below

(24) $$r_3(t_2) \leqslant r_2(t_2)$$

so that from (23), (24) we obtain

(25) $$r_2(t_2) \geqslant r_1(t_0),$$

from which by (17) we have

(26) $$c_2 \geq c_1.$$

The remaining statements in (18) follow from the fact that (20) and (21) have point (t_1, p_1) in common.

Viewing now $F_1(w)$, $w \leq t_1$ and $F_2(w)$, $w \geq t_1$ as parts of one distribution $F(w)$, we obtain for the failure rate of the latter from (8), (17), (18), and (26):

(27) $$r(w) = \begin{cases} 1 - e^{-c_1\Delta} & \text{if } w+\Delta < t \\ 1 - \dfrac{e^{-c_2(w+\Delta-d_2)}}{e^{-c_1(w-d_1)}} = 1 - \dfrac{e^{-c_2(\Delta-d_2)}}{e^{+c_1 d_1}}\, e^{-(c_2-c_1)w} & \text{if } w \leq t \text{ or } w+\Delta \geq t \\ 1 - e^{-c_2\Delta} & \text{if } w > t \\ \Delta > 0, \end{cases}$$

which is increasing in w, so that $F(w)$ is *IFR*.

From Lemma 1 it follows that no *IFR* distribution can pass through points (t_i, p_i), $i=0, 1, 2$ and above $F(w)$. Therefore indeed $\int_{t_0}^{t_2} F(w)dw$ is maximum among all *IFR* distributions.

IV. MAXIMIZATION OF OBJECTIVE FUNCTION

It is our intention to maximize objective function L given by (10) as

$$L = I \sum_{i=0}^{n-1} F^*(t_i) + a\left[\sum_{i=0}^{n-1} F^*(t_i)(t_{i+1}-t_i) - \int_0^{t_n} F^*(t)dt\right] + RF(T)$$

for a given checking policy $P=(t_0=0, t_1, t_2, \ldots, t_n=T)$ as defined in (11), by searching among all *IFR* distributions F which pass through the known point $(t_p=x, p)$, (7).

The form of the *IFR* distribution $F(w)$ leading to Max L is given, according to Lemma 2 by transformed exponential distribution functions (*t.e.d*) connecting all points $(t_i, p_i=F(t_i))$, $i=0, 1, 2, \ldots, n$ and (x, p). If m is the number of checks before x, we obtain the time sequence $t_0, t_1, t_2, \ldots, t_m, x, t_{m+1}, \ldots, t_n$, according to which above points are arranged. We recode this sequence as

(28) $$\{(t_{(i)}, p_{(i)}); \quad i=0, 1, \ldots, n+1\},$$

while

$$(t_0, p_0) = (0, 0) = (t_{(0)}, p_{(0)})$$

$$(x, p) = (t_{(m+1)}, p_{(m+1)}), \qquad c = c_{(m+1)}$$

$$(t_{n-1}, p_{n-1}) = (t_{(n)}, p_{(n)}), \qquad c_{n-1} = c_{(n)}$$

$$(t_n, p_n) = (T, 1) = (t_{(n+1)}, p_{(n+1)})$$

from which we obtain:

$$F(w)=\begin{cases}1-e^{-c(1)w} & \text{if } t_{(0)}=0\leq w\leq t_{(1)}\\ 1-e^{-c(i)(w-d(i))} & \text{if } t_{(i-1)}\leq w\leq t_{(i)},\ 1<i\leq n\\ 1 & \text{if } t_{(n)}<w\leq t_{(n+1)}=T\end{cases} \tag{29}$$

$$c_{(1)}<\alpha,\qquad c_{(1)}\leq c_{(2)}\leq\ \ldots\ \leq c_{(m+1)}\leq\ \ldots\ \leq c_{(n)},\qquad \alpha<c_{(m+1)}$$

$$c_{(i)}=\frac{1}{t_{(i)}-t_{(i-1)}}\ln\left(\frac{1-p_{(i-1)}}{1-p_{(i)}}\right),\qquad i=1,\ 2,\ \ldots,\ n$$

$$d_{(i)}=\frac{1}{c_{(i)}}\ln\left(1-p_{(i)}\right)+t_{(i)},$$

while we assumed

$$c_{(i)}=\begin{cases}c_i & \text{if } 1\leq i\leq m\\ c & \text{if } i=m+1\\ c_{i-1} & \text{if } m+2\leq i\leq n.\end{cases} \tag{30}$$

We observe that $F(w)$ passes through point $(0, 0)$ and possesses a jump from (t_{n-1}, p_{n-1}) to $(t_{n-1}{}^{+}, 1)$ (as allowed for an *IFR* distribution in [1], p. 25).

Objective function L, given feasible points (t_i, p_i), $i=1,\ 2,\ \ldots,\ n-1$, is indeed maximized by function $F(w)$ since $\int_0^{t_n=T}F^*(w)dw$ becomes a minimum and $RF(T)=R$ a maximum. Incidentally, because of the latter equality R does not affect the policy and will henceforth be assumed zero.

Maximization for given policy can now be carried out by dynamic programming methodology. The search extends over all $p_i=F(t_i)$ or equivalently over all $c_{(i)}$, $i=1,\ 2,\ \ldots,\ n$ since from (29) we obtain a one-to-one correspondence between both, as follows:

$$\begin{aligned}F^*(t_{(i)})&=F^*(t_{(i-1)})e^{-c\ (i)\ (t\ (i)-t\ (i-1))}\\ &=\prod_{j=1}^{i}e^{-c(j)(t(j)-t(j-1))},\qquad i=1,\ 2,\ \ldots,\ n.\end{aligned} \tag{31}$$

V. OPTIMIZATION BY DYNAMIC PROGRAMMING ALGORITHM

We define J_i, as loss on interval $(t_{i-1}, t_i]$:

$$J_i=J_i(t_{i-1},\ c_{i-1},\ t_i,\ c_i)=[I+a(t_i-t_{i-1})]F^*(t_{i-1})-a\int_{t_{i-1}}^{t_i}F^*(w)du\quad t_i<w\leq t_{i+1},\quad i=1,\ 2,\ \ldots,\ n.$$

If $J_i^* = \underset{F(w)}{\text{Max}}\, J_i$, we obtain from (29)

(33)

$$J_i^* = J_i^*(t_{i-1}, c_{i-1}, t_i, c_i) = \underset{F(w)}{\text{Max}}\, J_i(t_{i-1}, c_{i-1}, t_i, c_i)$$

$$= \begin{cases} [I + a(t_i - t_{i-1})]F^*(t_{i-1}) + a\,\dfrac{F^*(t_i) - F^*(t_{i-1})}{c_i} & \text{if } i = 1, 2, \ldots, \\ & m, m+2, \ldots, n-1 \\ [I + a(t_{m+1} - t_m)]F^*(t_m) + a\left[\dfrac{(1-p) - F^*(t_m)}{c} + \dfrac{F^*(t_{m+1}) - (1-p)}{c_{m+1}}\right] & \text{if } i = m+1 \\ [I + a(t_n - t_{n-1})]F^*(t_{n-1}) & \text{if } i = n, \end{cases}$$

where $c = c_{(m+1)}$ as given in (30) and $F^*(t_i)$ are functions of c, c_i and x, t_i only, according to (31).

Defining K_i as loss during time $(0, t_i]$ we obtain:

(34)
$$K_1(t_1, c_1) = J_1(t_0, c_0, t_1, c_1) = J_1(0, 0, t_1, c_1)$$

$$K_i(t_i, c_i) = K_{i-1}(t_{i-1}, c_{i-1}) + J_i(t_{i-1}, c_{i-1}, t_i, c_i)$$

$$= \sum_{j=1}^{i} J_j(t_{j-1}, c_{j-1}, t_j, c_j).$$

Let $K_n(T)$ signify the loss for the entire time period T. We therefore obtain:

(35)
$$K_n(T) = K_n(t_n, c_n)$$

$$= K_{n-1}(t_{n-1}, c_{n-1}) + J_n(t_{n-1}, c_{n-1}, t_n, c_n)$$

$$= \sum_{i=1}^{n} J_i(t_{i-1}, c_{i-1}, t_i, c_i)$$

$$= L$$

equality $K_n(T) = L$ can be verified by comparison of (32) and (10).

Optimization will now be carried out in two phases. In the first phase $K_{m+1}(t_{m+1}, c_{m+1})$ will be optimized, and in the second optimization of L will be achieved.

First Optimization Phase

Let the minimax optimized loss for time period $(0, t_i]$ be

(36)
$$K_i^*(t_i, c_i) = \underset{P}{\text{Min}}\ \underset{F}{\text{Max}}\, K_i(t_i, c_i),$$

where P is the checking policy (11) and F is distribution (29). By using dynamic programming procedure we then obtain

$$K_0^* = 0 \tag{37}$$

$$K_1^*(t_1, c_1) = J_1^*(0, 0, t_1, c_1) = I + at_1 + a\frac{F^*(t_1) - 1}{c_1}, \qquad c_1 < \alpha$$
$$0 < t_1 \leq x$$

$$K_2^*(t_2, c_2) = \operatorname*{Min}_{0 < t_1 < t_2} \operatorname*{Max}_{0 < c_1 \leq \operatorname{Min}(c_2, \alpha)} \{K_1^*(t_1, c_1) + J_2^*(t_1, c_1, t_2, c_2)\}$$
$$0 < t_2 \leq x$$

$$K_i^*(t_i, c_i) = \operatorname*{Min}_{0 < t_{i-1} < t_i} \operatorname*{Max}_{0 < c_{i-1} \leq c_i} \{K_{i-1}^*(t_{i-1}, c_{i-1}) + J_i^*(t_{i-1}, c_{i-1}, t_i, c_i)\} \qquad i = 3, 4, \ldots, m$$
$$0 < t_i \leq x$$

$$K_{m+1}^*(t_{m+1}, c_{m+1}) = \operatorname*{Min}_{0 < t_m < x} \operatorname*{Max}_{0 < c_m \leq c \leq c_{m+1}} \{K_m^*(t_m, c_m) + J_{m+1}^*(t_m, c_m, t_{m+1}, c_{m+1})\} \qquad m = 0, 1, 2, \ldots .$$
$$x \leq t_{m+1} \leq T$$

Since at point (x, p), $F(w)$ crosses the exponential distribution $F(u)$ (see (15)) from below, the result is $\alpha < c \leq c_{m+1}$, as shown already in (29).

We now chose for each (t_{m+1}, c_{m+1}) that $m = m^*$ which renders K_{m+1}^* a minimum. The search is simplified by assuming K_{m+1}^* unimodal in m. We obtain

$$K_{m^*+1}^{**}(t_{m^*+1}, c_{m^*+1}) = \operatorname*{Min}_{m=0,1,2,} \{K_{m+1}^*(t_{m+1}, c_{m+1})\}. \tag{38}$$

Second Optimization Phase

We continue our iterative procedure, and based on previous results we obtain the general term:

$$K_{m^*+i}^{**}(t_{m^*+i}, c_{m^*+i}) = \operatorname*{Min}_{x \leq t_{m^*+i-1} \leq t_{m^*+i}} \operatorname*{Max}_{\alpha \leq c_{m^*+i-1} \leq c_{m^*+i}} \{K_{m^*+i-1}^{**}(t_{m^*+i-1}, c_{m^*+i-1}) + J_{m^*+i}^*(t_{m^*+i-1}, c_{m^*+i-1}, t_{m^*+i}, c_{m^*+i})\} \qquad i = 2, 3, 4, \ldots , \tag{39}$$
$$x \leq t_{m^*+i} \leq T$$

and, in particular, the term for the total optimal loss with $n = m^* + i$ checks, is derived from (38) and (33) as:

$$K_n^{**}(T) = K_{n=m^*+i}^{**}(t_{n=m^*+i}, c_{n=m^*+i}), \qquad i = 2, 3, 4, \ldots . \tag{40}$$

Assuming unimodality in i or, respectively, in $n = m^* + i$, the optimal loss L^{**} at $n^* = m^* + i^*$ is:

$$L^{**} = \operatorname*{Min}_{i=2,3,\ldots} \{K_{n=m^*+i}^{**}(T)\}. \tag{41}$$

The optimal policy $P^{**} = \{t_1^*, t_2^*, \ldots, t_n^*\}$ is finally derived by a backward recursive procedure from computers memory.

VI. NUMERICAL RESULTS AND CONCLUSION

Several numerical results are presented in Table 1, while a normalized parameter vector W_n

$= \{I = 1, a, R = 0, T = 2, x, p\}$ (see (12)) is utilized. According to notation of Sect. I (repeated below Table 1), row $L^*(U)$ gives losses sustained if the minimax *IFR* policy is applied to the [0, 2] uniform distribution. As expected from (6) it can be observed that $L^*(U) < L^{**}$ for all cases in which (x, p) is a point on the uniform distribution function. Several examples employ cost $a = 25$ while (x, p) is changed. Further rows give: $L_0(U)$ — the minimum loss assuming knowledge of the uniform distribution [0, 2], L_1 — the minimax loss if point (x, p) is given, but distribution is unknown, and L_2 — the minimax loss if the distribution is entirely unspecified, (for L_1 and L_2 any distribution and not only *IFR* as for L^{**}, are considered). The intuitively appealing result $L_0(U) < L^{**} < L_1 < L_2$ is verified.

TABLE 1. *Model* L *Cost and policies. Comparison of different models.*

Costs: $I = 1$, $T = 2$, $R = 0$.

a	5	10	25	25	25	25	25	50
x	1	1	1.4	1	0.8	0.6	1	1
p	0.5	0.5	0.7	0.5	0.4	0.3	0.9	0.5
$L_0(U)$	3.500	–	7.175	7.175	7.175	7.175	7.175	9.940
L^{**}	–	5.114	8.808	7.647	7.571	7.999	7.254	11.108
$L^*(U)$		4.777	8.760	7.186	7.352	7.345	7.843	10.268
L_1	4.833	–	10.041	10.160	–	10.282	–	–
L_2	5.000	–	10.500	10.500	10.500	10.500	10.500	14.642
t_1		0.619	0.667	0.333	0.270	0.457	0.333	0.333
t_2		1.000	1.333	0.667	0.541	0.747	0.667	0.667
t_3		1.368	1.463	1.000	0.800	1.042	1.000	1.000
t_4		1.689	1.558	1.316	1.053	1.337	1.105	1.158
t_5		1.947	1.653	1.579	1.305	1.632	1.211	1.316
t_6		2.000	1.747	1.789	1.558	1.853	1.316	1.474
t_7			1.842	1.947	1.747	1.926	1.421	1.632
t_8			1.905	2.000	1.937	2.000	1.526	1.789
t_9			1.968		2.000		1.632	1.947
t_{10}			2.000				1.737	2.000
n		6	10	8	9	8	13	10

$L_0(U)$ — Uniform distribution — optimal policy
L^{**} — Unknown distribution, (x, p) known — minimax IFR policy
$L^*(U)$ — Uniform distribution — minimax IFR policy
L_1 — Unknown distribution, (x, p) known — minimax policy
L_2 — Unknown distribution — minimax policy

The numerical table is used for the following example.

EXAMPLE: A certain plant disease is liable to appear during a period of 20 days each year. The disease can only be detected by inspection, which costs \$200. Each day passing between infection to detection causes damage of \$500 per day. After detection a treatment of \$1,000 is needed. Since the specific disease never appears a second time during the same year, the model of this study is applicable. The distribution of time to infection is supposed to be *IFR* with about 50 percent chance that infection takes place before 10 days have passed.

The parameters can be expressed in the following form, consistent with Table 1, column 4 and formulae (12) to (14).

$$\begin{aligned} T &= \text{two units of 10 days each} \\ I &= \$200 \text{ per check} \\ a &= \$5{,}000 = \$25 \times 200 \text{ per 10 days} \\ R &= \$1{,}000 \\ (x, p) &= (10 \text{ days}, 0.5) \end{aligned}$$

from which we obtain: the average total (maximum) loss per year

$$L^{**} = 7.647 \times 200 + 1000 = \$2{,}529.4$$

the checking time sequence is

$$3, 7, 10, 13, 16, 18, 19, 20 \text{ days}.$$

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THE ECONOMIC EFFECTS OF INSPECTOR ERROR ON ATTRIBUTE SAMPLING PLANS

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ABSTRACT

In this paper the effects of inspector error on a cost-based quality control system are investigated. The system examined is of a single sampling plan design involving several cost components. Both type I and type II inspector errors are considered. The model employs a process distribution, thus assuming that a stochastic process of some kind governs the quality of incoming lots. Optimal plan design is investigated under both error-free and error-prone inspection procedures and some comparisons are made.

INTRODUCTION

Sampling inspection plans provide decision-making procedures for controlling and improving lot quality through accept-reject decisions. Classically, sampling inspection plans have assumed that the inspection of an item is perfect, with no errors in judgement being made by the inspector. This assumption is, or course, false. In reality an inspector, whether human or machine, is capable of making two types of errors—classifying a good item as bad (type I error) or classifying a bad item as good (type II error). Conclusive evidence exists verifying this fact. See, for example, any one of References [8] through [14].

Numerous authors have recognized the presence of inspection error and have incorporated proper allowances into their sampling plan formulations. Noteworthy of mention is the work conducted by Minton [16], Davis [4], McKnight [15], and Ayoub, et al. [1], [2]. These authors discuss the effects of inspection error on several measures of performance. They consider average outgoing quality (*AOQ*) when items which appear defective are replaced; however, they assume perfect inspection of replacements. They also discuss the performance measure average total inspection (*ATI*) when items which appear to be defective are removed from the sampled lot but are not replaced. Later, Collins, et al. [3] relax the assumption of perfect inspection of replacements for *AOQ* and allow for defective replacements in the formulation of *ATI*.

To date, the effects of inspection error have been considered mainly in sampling plan designs which are based on purely statistical measures. The effects of such errors on economic sampling plan designs has received little attention. Since economic designs are receiving widespread acceptance, it seems natural to investigate the effects of inspector errors on these plans. In this paper we consider their effect on the economic design of single sampling attribute inspection plans.

Notation

Consider the following symbols:

N = lot size,

n = sample size,

c = acceptance number,

x = actual number of defectives in the n items sampled,

X = actual number of defectives in the lot,

$u = X - x$ = number of defectives remaining in the unsampled portion of the lot,

x_e = apparent number of defectives in the n items sampled,

a_1 = action representing lot acceptance

a_2 = action representing lot rejection

e_1 = probability of a type I inspection error,

e_2 = probability of a type II inspection error,

c_i = cost of inspecting an item,

c_v = amount invested in each item,

c_r = cost of reworking or repairing a defective item, and

c_a = cost attributed to a defective item in an accepted lot.

Wortham and Mogg [17] report nine possible rectification schemes which can be considered in determining average outgoing quality (AOQ). Six of these same possibilities were used by Wortham and Wilson [18] for the design of optimal sequential sampling plans. These same six possibilities will be used in this paper. Consider the following three sample dispositions and two rejected lot dispositions [18].

Sample:

i) discard the entire sample,

ii) replace, repair, or rework all defectives, or

iii) only enumerate defectives.

Rejected lots:

i) discard the entire lot, or

ii) replace, repair, or rework all defectives.

Let $T_{a_1}(n, x, u)$ be the cost of the decision to accept a lot containing u defectives after n items are sampled and x defectives are found. Let $T_{a_2}(n, x, u)$ be the cost of the decision to reject the lot.

Table I shows the variety of models which arise from the six possible disposition policies.

MODEL FORMULATION WITHOUT ERROR

For each of the above six disposition possibilities there are two courses of action, either to accept the lot, action a_1, or to reject it, action a_2. The action to accept the lot is taken if the number of defectives, x, in a sample of size n is such that $x \leq c$; otherwise the decision to reject the lot is made. In our design we seek the values of the decision variables n and c which minimize the expected total cost of quality control per lot. That is, we seek the values of n and c which minimize the total cost equation

$$C_T(n) = E_x \min_a E_{u|x} T_a(n, x, u) , \tag{1}$$

TABLE I. *No inspection error*

Sample dispositions	Lot dispositions	
	(i)	(ii)
(i)	$T_{a_1}=(c_v+c_i)n+c_a u$	$T_{a_1}=(c_v+c_i)n+c_a u$
	$T_{a_2}=c_i n+c_v N$	$T_{a_2}=c_i N+c_v n+c_r u$
(ii)	$T_{a_1}=c_i n+c_r x+c_a u$	$T_{a_1}=c_i n+c_r x+c_a u$
	$T_{a_2}=c_i n+c_r x+c_v N$	$T_{a_2}=c_i N+c_r x+c_r u$
(iii)	$T_{a_1}=c_i n+c_a x+c_a u$	$T_{a_1}=c_i n+c_a x+c_a u$
	$T_{a_2}=c_i n+c_v N$	$T_{a_2}=c_i N+c_r x+c_r u$

where a represents the action to be taken, $E_{u|x}$ indicates that the expectation is with respect to the distribution of the number of defectives in the unsampled portion of the lot, u, given x, and E_x indicates that expectation is with respect to the unconditional distribution of the number of defectives in the sample, x.

For each disposition policy listed in Table I it is observed that the costs can be written as a linear function of the number of defectives in the uninspected portion of the lot, u. That is,

$$T_{a_i}(n, x, u) = K_i + C_i u\,, \tag{2}$$

where K_i and C_i are constants not involving u. Thus Equation (1) can be written as

$$\begin{aligned} C_T(n) &= E_x \min_a E_{u|x}[K_i + C_i u] \\ &= E_x \min_a [K_i + C_i\, E(u|x)]\,, \end{aligned} \tag{3}$$

where $E(u|x)$ is the expected number of defectives in the unsampled portion of the lot given that x defectives were found in the sampled portion.

Now consider the terminal situation in which x defectives are observed in a sample containing n items. From Equation (3) it can be seen that the optimal decision would be to accept the lot if

$$K_1 + C_1\, E(u|x) \leq K_2 + C_2\, E(u|x)\,, \tag{4}$$

otherwise, the optimal decision is to reject the lot. In many cases of practical significance we find $E(u|x)$ to be a linear function of the number of defectives in the sample, x. In these situations an expression $\phi(\cdot)$ satisfying the relation

$$x \leq \phi\ (n, N, K_1, K_2, C_1, C_2, \underset{\sim}{p}) \tag{5}$$

can be obtained from Equation (4), where $\underset{\sim}{p}$ denotes the vector of parameters comprising the conditional distribution of the number of defectives in the unsampled portion of the lot, $h(u|x)$. In such

cases the largest integer less than or equal to $\phi(\cdot)$ will be the optimal choice for the acceptance number c.

Once the acceptance number c has been obtained as a function of n, we may then proceed to find the optimal sample size. This is best accomplished through the use of a digital computer. Any standard integer search procedure can be used to find the value of n which minimizes the expected total cost equation

$$C_T(n) = \sum_{x=0}^{c} [K_1 + C_1 E(u|x)]g_n(x) + \sum_{x=c+1}^{n} [K_2 + C_2 E(u|x)]g_n(x) , \tag{6}$$

where $g_n(x)$ is the unconditional distribution of the number of defectives in the sample.

Distributional Considerations

Let $f_n(x|X)$ be the distribution of defectives, x, in a sample of size n given X defectives in the lot and let $f_N(X)$ be the distribution of defectives in a lot of size N. It follows that $g_n(x)$ is given by

$$g_n(x) = \sum_{X=x}^{N-n-x} f_n(x|X) \cdot f_N(X) . \tag{7}$$

If sampling is conducted without replacement, the conditional distribution $f_n(x|X)$ is hypergeometric. Thus

$$f_n(x|X) = f_n(x|x+u) = \binom{n}{x}\binom{N-n}{u} \Big/ \binom{N}{x+u} \tag{8}$$

and

$$g_n(x) = \sum_{u=0}^{N-n} f_n(x|u+x) \cdot f_N(u+x). \tag{9}$$

Now by invoking Bayes Theorem, the distribution of defectives in the unsampled portion of the lot, u, given x, can be expressed as

$$h(u|x) = f_n(x|u+x) \cdot f_N(u+x)/g_n(x) = \binom{n}{x}\binom{N-n}{u} f_N(u+x) \Big/ \binom{N}{x+u} g_n(x). \tag{10}$$

Wortham and Wilson (18) have shown that the expectation of u given x, $E(u|x)$, can be expressed conveniently as

$$E(u|x) = \sum_{u=0}^{N-n} uh(u|x) = \frac{(N-n)\ (x+1)}{n+1} \cdot \frac{g_{n+1}(x+1)}{g_n(x)} \tag{11}$$

where $g_{n+1}(x+1)$ denotes the unconditional distribution of $x+1$ based upon a sample of $n+1$ items. This expression provides a very useful relationship. Specifically, it enables the calculation of $E(u|x)$ in terms of the unconditional distribution of the number of defectives in the sample.

In addition, Hald [7] makes an observation which can further simplify the calculation of $E(u|x)$. He notes that if the prior distribution, $f_N(X)$, is of the hypergeometric, binomial, polya, rectangular, or mixed binomial families, the unconditional distribution $g_n(x)$ will be of the same family as the prior. This general property is termed reproducibility. Thus in the case of reproducibility $g_n(x)$ assumes the same form as $f_N(X)$, the only difference being that n and x have replaced N and X. To illustrate these concepts let us consider the following example.

EXAMPLE I: Consider the disposition policy (ii, ii) where defectives in both samples and rejected lots are replaced. The cost terms for this policy can be found in Table I. They are

$$T_{a_1}(n, x, u) = K_1 + C_1 u$$

and

$$T_{a_2}(n, x, u) = K_2 + C_2 u,$$

where

(12)
$$K_1 = c_i n + c_r x, \qquad C_1 = c_a,$$

$$K_2 = c_i N + c_r x, \qquad C_2 = c_r.$$

If sampling without replacement is employed and if the number of defectives in a lot of size N can be described by the Polya distribution

(13)
$$f_N(X) = \binom{N}{X} \frac{\Gamma(x' + X)\Gamma(n' + N - x' - X)\Gamma(n')}{\Gamma(x')\Gamma(n' - x')\Gamma(n' + n)}$$

with parameters x' and n' then by the property of reproducibility we have

(14)
$$g_n(x) = \binom{n}{x} \frac{\Gamma(x' + x)\Gamma(n' + n - x' - x)\Gamma(n')}{\Gamma(x')\Gamma(n' - x')\Gamma(n' + n)}.$$

Substituting this relationship into Equation (11) yields

(15)
$$E(u|x) = (N - n)\left(\frac{\bar{l} + x\gamma}{1 + n\gamma}\right),$$

where

$$\bar{l} = \frac{x'}{n'} \qquad \text{and} \qquad \gamma = \frac{1}{n'}.$$

Now for a fixed value of n, we have from Equation (4) that

(16)
$$c_i n + c_r x + c_a E(u|x) \leq c_i N + c_r x + c_r E(u|x)$$

$$E(u|x) \leq \frac{c_i(N - n)}{(c_a - c_r)}$$

$$(N-n)\left(\frac{\bar{l}+x\gamma}{1+n\gamma}\right) \leq \frac{c_i(N-n)}{(c_a-c_r)}$$

$$\bar{l}+x\gamma \leq \frac{c_i(1+n\gamma)}{(c_a-c_r)}$$

$$x \leq \frac{c_i(1+n\gamma)}{(c_a-c_r)\gamma} - \frac{\bar{l}}{\gamma}.$$

Thus the optimal acceptance number as a function of the sample size n is given by

$$c = \left[\frac{c_i(1+n\gamma)}{(c_a-c_r)\gamma} - \frac{\bar{l}}{\gamma}\right], \tag{17}$$

where $[y]$ denotes the largest integer less than or equal to y. Of course, it is understood that when $[y]$ is negative c should be set equal to zero.

For the specific cost and parameter values

$$c_i = \$2.00 \qquad x' = 3$$
$$c_r = \$1.90 \qquad n' = 60$$
$$c_a = \$40.00$$

we obtain

$$c = [(2.85+n)/19.05]. \tag{18}$$

Proper substitution of these values into Equation (6) followed by the optimization of $C_T(n)$ over n yields the minimum cost sampling plan ($n=126$, $c=6$) at an expected total cost per lot of \$1,744.87. This plan was obtained via a computer program written in FORTRAN IV. The program was executived on an IBM 370/155 and required 5 seconds of run time. A simple incremental search procedure was used to find the optimal value of n.

MODEL FORMULATION WITH ERROR

Let us now consider the situation in which type I and type II errors enter into the inspection process. In this case an expression for the apparent number of defectives in both the sample and the lot must be defined. The apparent number of defectives is the number of defectives as it appears to the inspector.

Let

e_1 = the probability that a good item is classified as defective (type I error),

and

e_2 = the probability that a defective item is classified as good (type II error).

Then the expected apparent number of defectives in the sample, x_e, is given by

$$x_e = (n-x)e_1 + (1-e_2)x \tag{19}$$

$$= ne_1 + \epsilon x,$$

and the expected apparent number of defectives in the lot, X_e, by

$$X_e = (N-X)e_1 + (1-e_2)X \tag{20}$$
$$= Ne_1 + \epsilon X,$$

where ϵ is the probability that an item is classed correctly and is given by

$$\epsilon = 1 - e_1 - e_2 .$$

Depending upon the disposition policy, the number of defectives remaining in a lot may consist of some or all of the following terms:

ue_2 = the number of defective items classed as good in the screened portion of a lot,
$u(1-e_2)$ = the number of defective items classed as defective in the screened portion of a lot,
$(N-n-u)e_1$ = the number of good items classed as bad during lot rectification,
xe_2 = the number of defective items classed as good in the sample.

Using these terms we may extend the results of Table I to the case where inspection errors are present in the inspection process. Table II shows the variety of models which arise when inspection error is present. Note that it has been assumed that replacement items are error free. Such would be the case if items used for replacement had been, without question, reworked.

If for each disposition policy listed in Table II, x_e is replaced by Equation (19) and X_e is replaced by Equation (20) then a more suitable representation for each disposition policy can be obtained. These results are displayed in Table III. Note that in this formulation X has been replaced by $u + x$. This substitution allows each policy to be represented solely in terms of the random variables u and x. Furthermore, as we shall later observe, this representation allows us to visualize the net effect inspection error has on the per unit costs c_i, c_v, c_r, and c_a.

TABLE II. *Inspection error in terms of* x_e *and* X_e

Sample dispositions	Lot dispositions	
	(i)	(ii)
(i)	$T'_{a_1} = (c_v + c_i)n + c_a u$	$T'_{a_1} = (c_v + c_i)n + c_a u$
	$T'_{a_2} = c_i n + c_v N$	$T'_{a_2} = c_i N + c_v n + c_r[u(1-e_2) + (N-n-u)e_1] + c_a e_2 u$
(ii)	$T'_{a_1} = c_i n + c_r x_e + c_a x e_2 + c_a u$	$T'_{a_1} = c_i n + c_r x_e + c_a x e_2 + c_a u$
	$T'_{a_2} = c_i n + c_r x_e + c_v N$	$T'_{a_2} = c_i N + c_r X_e + c_a x e_2 + c_a u e_2$
(iii)	$T'_{a_1} = c_i n + c_a x + c_a u$	$T'_{a_1} = c_i n + c_a x + c_a u$
	$T'_{a_2} = c_i n + c_v N$	$T'_{a_2} = c_i N + c_r X_e + c_a x e_2 + c_a u e_2$

TABLE III. *Inspection error in terms of* u *and* x

Sample dispositions	Lot dispositions	
	(i)	(ii)
(i)	$T'_{a_1} = (c_v + c_i)n + c_a u$	$T'_{a_1} = (c_v + c_i)n + c_a u$
	$T'_{a_2} = c_i n + c_v N$	$T'_{a_2} = (c_i + c_r e_1)N + (c_v - c_r e_1)n + (c_r \epsilon + c_a e_2)u$
(ii)	$T'_{a_1} = (c_i + c_r e_1)n + (c_r \epsilon + c_a e_2)x + c_a u$	$T'_{a_1} = (c_i + c_r e_1)n + (c_r \epsilon + c_a e_2)x + c_a u$
	$T'_{a_2} = (c_i + c_r e_1)n + c_r \epsilon x + c_v N$	$T'_{a_2} = (c_i + c_r e_1)N + (c_r \epsilon + c_a e_2)x + (c_r \epsilon + c_a e_2)u$
(iii)	$T'_{a_1} = c_i n + c_a x + c_a u$	$T'_{a_1} = c_i n + c_a x + c_a u$
	$T'_{a_2} = c_i n + c_v N$	$T'_{a_2} = (c_i + c_r e_1)N + (c_r \epsilon + c_a e_2)x + (c_r \epsilon + c_a e_2)u$

It is observed that each of the cost terms associated with the disposition policies listed in Table III is a linear function of u and as such can be expressed in the form

$$T'_{a_i} = K'_i + C'_i\, u, \tag{21}$$

where K'_i and C'_i are constants not involving u. Thus when error is present in the inspection process the expected total cost is given by

$$C'_T(n) = E_x \min_a E_{u|x}\,[K'_i + C'_i\, u] = E_x \min_a [K'_i + C'_i\, E(u|x)]. \tag{22}$$

Note that Equation (22) is of the same form as Equation (3) and as such may be expressed as follows:

$$C'_T(n) = \sum_{x=0}^{c'} [K'_1 + C'_1 E(u|x)]g_n(x) + \sum_{x=c'+1}^{n} [K'_2 + C'_2 E(u|x)]g_n(x). \tag{23}$$

The value of c' may be obtained in a manner completely analogous to the development preceding Equation (5). Thus under the assumption that $E(u|x)$ is a linear function of u, c' is the largest integer less than or equal to $\phi(\cdot)$ where $\phi(\cdot)$ is now defined by the relationship

$$x \leqslant \phi(n, N, K'_1, K'_2, C'_1, C'_2, \underset{\sim}{p}). \tag{24}$$

Recall that x_e, not x, is the number of defectives that the inspector actually observes. Thus when inspection error is present the acceptance number, c, is found by replacing x in relation (24) by its equivalent $(x_e - ne_1)/\epsilon$ and solving for x_e. Doing so, we obtain

$$x_e \leqslant \phi(\cdot)\epsilon + ne_1. \tag{25}$$

Therefore the acceptance number supplied to the inspector is simply the largest integer less than or equal to the right-hand side of relation (25).

EXAMPLE II: Consider the problem as formulated in Example I. When inspection error is present we have from Table III that

$$T'_{a_1}(n, x, u) = K'_1 + C'_1 u$$

and

$$T'_{a_2}(n, x, u) = K'_2 + C'_2 u,$$

where

$$K'_1 = (c_i + c_r e_1) n + (c_r \epsilon + c_a e_2) x \qquad C'_1 = c_a \tag{26}$$

$$K'_2 = (c_i + c_r e_1) N + (c_r \epsilon + c_a e_2) x \qquad C'_2 = c_a e_2 + c_r \epsilon .$$

Replacing K_i and C_i in Equation (4) by K'_i and C'_i and solving for $E(u|x)$, we obtain

$$E(u|x) \leqslant \frac{(c_i + c_r e_1)\ (N - n)}{c_a(1 - e_2) - c_r \epsilon}.$$

This relation can be further simplified by substituting Equation (15) for $E(u|x)$ and solving for x to obtain

$$x \leqslant \frac{(c_i + c_r e_1)\ (1 + n\gamma)}{[c_a(1 - e_2) - c_r \epsilon]\gamma} - \frac{\bar{l}}{\gamma}. \tag{27}$$

Thus, when inspection error is present the value of c' is the largest integer less than or equal to the right-hand side of relation (27). If we denote this expression by ϕ then the acceptance number, c, is the largest integer less than or equal to the quantity $\phi\epsilon + ne_1$.

At this point it is interesting to note that if we make the substitutions

$$c_i = [c_i + c_r e_1]$$

and

$$c_r = c_r \epsilon + c_a e_2$$

into the cost terms for the disposition policy (ii, ii) of Table I we obtain the cost terms for this same disposition policy when inspection error is present, Table III. In other words, for this disposition policy, inspection error has the net effect of increasing the cost of inspection per unit and the cost of rejection per unit by the amounts $c_r e_1$, and $c_r \epsilon + c_a e_2$, respectively. For example, if we adopt the costs of Example 1, and assume errors of $(e_1 = 0.10, e_2 = 0.15)$ then the per unit costs of inspection and replacement become \$2.119 and \$7.425 respectively, a per unit increase of \$0.119 for inspection and \$5.525 for replacement.

By the use of the cost and distributional parameters of Example I, Equation (23) was evaluated at the optimal plan $(n = 126, c = 6)$ for various combinations of inspector errors (e_1, e_2). For each evaluation the value of c' was given by

$$c' = (6.76 - 126\, e_1)/\epsilon,$$

where 6.76 is the right-hand side of relation (16) when evaluated at the given parameters. The results of these evaluations are listed in Table IV. These results show the effects of using the error-free sampling plan ($n=126$, $c=6$) when inspection error is present. Note that for several error pairs the effects are substantial. For example, if the inspection process includes errors of ($e_1=0.10$, $e_2=0.15$) then the true expected total cost is \$2,513.54, not \$1,744.87 as one would be led to believe if he was unaware of existing inspection error. Also listed is the percentage error. For the error pair ($e_1=0.10$, $e_2=0.15$) the percentage error is

$$\frac{\$2{,}513.54 - \$1{,}744.87}{\$1{,}744.87} \times 100 = 44.05\%$$

TABLE IV. *Expected cost of the optimal sampling plan* (n = 126, c = 6) *for selected error pairs*

(e_1, e_2)	Cost of acceptance	Cost of rejection	Total cost C'_{TE}	Percentage error Δ_1
(0,.01)	\$870.51	\$886.56	\$1,757.07	0.70
(0,.03)	872.07	909.40	1,781.47	2.10
(0,.05)	1,049.36	752.67	1,802.03	3.28
(0,.10)	1,054.35	801.41	1,855.76	6.36
(0,.15)	1,059.33	850.16	1,909.49	9.43
(.01,0)	689.27	1,078.34	1,767.61	1.30
(.01,.01)	689.84	1,091.47	1,781.31	2.09
(.01,.03)	690.98	1,117.73	1,808.71	3.66
(.01,.05)	875.01	939.45	1,814.46	3.99
(.01,.10)	878.90	996.57	1,875.47	7.48
(.01,.15)	882.80	1,053.69	1,936.49	10.98
(.03,0)	342.98	1,539.40	1,882.38	7.88
(.03,.01)	343.19	1,555.68	1,898.87	8.83
(.03,.03)	512.13	1,358.09	1,870.22	7.18
(.03,.05)	512.88	1,387.66	1,900.54	8.92
(.03,.10)	514.74	1,461.59	1,976.33	13.27
(.03,.15)	516.60	1,535.53	2,052.13	17.61
(.05,0)	92.05	1,978.54	2,070.59	18.67
(.05,.01)	92.07	1,996.91	2,088.98	19.72
(.05,.03)	92.12	2,033.67	2,125.79	21.83
(.05,.05)	92.17	2,070.42	2,162.59	23.94
(.05,.10)	201.32	1,961.45	2,162.77	23.95
(.05,.15)	201.79	2,048.97	2,250.76	28.99
(.10,0)	26.28	2,204.28	2,230.56	27.84
(.10,.01)	26.28	2,223.14	2,249.42	28.92
(.10,.03)	26.28	2,260.87	2,287.15	31.08
(.10,.05)	26.28	2,298.61	2,324.89	33.24
(.10,.10)	26.28	2,392.93	2,419.21	38.65
(.10,.15)	26.28	2,487.26	2,513.54	44.05
(.15,0)	26.65	2,291.52	2,318.17	32.86
(.15,.01)	26.65	2,310.39	2,337.04	33.94
(.15,.03)	26.65	2,348.12	2,374.77	36.10
(.15,.05)	26.65	2,385.85	2,412.50	38.26
(.15,.10)	26.65	2,480.18	2,506.83	43.67
(.15,.15)	26.65	2,574.51	2,601.16	49.07

In general, the percentage error is defined by

$$\Delta_1 = \frac{(C'_{TE} - C_T)}{C_T} \times 100,$$

where

C_T = \$1,744.87, the expected total cost of the optimal sampling plan when the inspection process is error-free,

and

C'_{TE} = the expected total cost of the optimal sampling plan under the assumption that the inspection process is error-free, when inspection error is present.

Table V lists the results obtained from optimizing Equation (23). These results give the optimal sampling plan and its expected cost when inspection error is included into the design. Also provided

TABLE V. *Optimal sampling plan for selected error pairs*

(e_1, e_2)	Optimal sample plan (n, c)	Cost of acceptance	Cost of rejection	Total cost C'_T	Percentage error Δ_2
(0,.01)	(117,6)	\$958.52	\$797.97	\$1,756.49	0.67
(0,.03)	(117,6)	960.15	819.07	1,779.22	1.97
(0,.05)	(110,6)	1,036.46	764.68	1,801.14	3.22
(0,.10)	(101,5)	1,145.07	707.57	1,852.64	6.18
(0,.15)	(77,4)	1,265.69	632.47	1,898.16	8.79
(.01,0)	(117.7)	959.07	793.88	1,752.95	0.46
(.01,.01)	(117,7)	959.88	804.43	1,764.31	1.11
(.01,.03)	(115,7)	982.38	804.58	1,786.96	2.41
(.01,.05)	(110,7)	1,037.81	770.52	1,808.33	3.64
(.01,.10)	(101,6)	1,146.39	712.61	1,859.00	6.54
(.01,.15)	(77,5)	1,266.76	636.69	1,903.45	9.09
(.03,0)	(117,9)	961.80	806.79	1,768.59	1.36
(.03,.01)	(115,9)	983.45	796.47	1,779.92	2.01
(.03,.03)	(110,9)	1,038.82	762.62	1,801.44	3.24
(.03,.05)	(109,9)	1,051.63	770.99	1,822.62	4.46
(.03,.10)	(84,7)	1,163.03	708.52	1,871.55	7.26
(.03,.15)	(76,6)	1,283.99	629.83	1,913.82	9.68
(.05,0)	(110,11)	1,038.99	744.93	1,783.92	2.24
(.05,.01)	(110,11)	1,039.84	754.72	1,794.56	2.85
(.05,.03)	(109,11)	1,052.63	763.14	1,815.77	4.06
(0.5,.05)	(105,11)	1,100.05	736.24	1,836.29	5.42
(.05,.10)	(81,8)	1,208.34	674.79	1,883.13	7.92
(.05,.15)	(71,7)	1,363.56	560.25	1,923.81	10.26
(.10,0)	(105,16)	1,102.43	717.11	1,819.54	4.28
(.10,.01)	(103,16)	1,126.74	702.88	1,829.62	4.86
(.10,.03)	(101,15)	1,152.17	696.73	1,848.90	5.96
(.10,.05)	(81,12)	1,210.16	657.54	1,867.70	7.04
(.10,.10)	(76,12)	1,287.75	621.88	1,909.63	9.44
(.10,.15)	(64,10)	1,479.96	465.82	1,945.78	11.51
(.15,0)	(101,20)	1,156.15	695.69	1,851.84	6.13
(.15,.01)	(81,16)	1,212.71	648.38	1,861.09	6.66
(.15,.03)	(77,15)	1,272.92	605.76	1,878.68	7.67
(.15,.05)	(76,15)	1,289.38	605.71	1,895.09	8.61
(.15,.10)	(69,14)	1,401.55	531.43	1,932.98	10.78
(.15,.15)	(62,13)	1,517.09	447.45	1,964.54	12.59

TABLE VI. *Percentage error realized by using plan* (n = 126, c = 6) *instead of the optimal plan when inspection error is present*

(e_1, e_2)	Total cost C'_{TE}	Total cost C'_T	Percentage error Δ_3
(0,.01)	$1,757.07	$1,756.49	0.03
(0,.03)	1,781.47	1,779.22	0.13
(0,.05)	1,802.03	1,801.14	0.05
(0,.10)	1,855.76	1,852.64	0.17
(0,.15)	1,909.49	1,898.16	0.60
(.01,0)	1,767.61	1,752.95	0.84
(.01,.01)	1,781.31	1,764.31	0.96
(.01,.03)	1,808.71	1,786.96	1.22
(.01,.05)	1,814.46	1,808.33	0.34
(.01,.10)	1,875.47	1,859.00	0.89
(.01,.15)	1,936.49	1,903.45	1.74
(.03,0)	1,882.38	1,768.59	6.43
(.03,.01)	1,898.87	1,779.92	6.68
(.03,.03)	1,870.22	1,801.44	3.82
(.03,.05)	1,900.54	1,822.62	4.28
(.03,.10)	1,976.33	1,871.55	5.60
(.03,.15)	2,052.13	1,913.82	7.23
(.05,0)	2,070.59	1,783.92	16.07
(.05,.01)	2,088.98	1,794.56	16.41
(.05,.03)	2,125.79	1,815.77	17.07
(.05,.05)	2,162.59	1,836.29	17.77
(.05,.10)	2,162.77	1,883.13	14.85
(.05,.15)	2,250.76	1,923.81	16.99
(.10,0)	2,230.56	1,819.54	22.59
(.10,.01)	2,249.42	1,829.62	22.94
(.10,.03)	2,287.15	1,848.90	23.70
(.10,.05)	2,324.89	1,867.70	24.48
(.10,.10)	2,419.21	1,909.63	26.68
(.10,.15)	2,513.54	1,945.78	29.18
(.15,0)	2,318.17	1,851.84	25.18
(.15,.01)	2,337.04	1,861.09	25.57
(.15,.03)	2,374.77	1,878.68	26.41
(.15,.05)	2,412.50	1,895.09	27.30
(.15,.10)	2,506.83	1,932.98	26.69
(.15,.15)	2,601.16	1,964.54	32.41

is a measure of the error one suffers by having an inspection process which is not error free. This result is expressed as a relative percentage and is defined by the relation

$$\Delta_2 = \frac{(C'_T - C_T)}{C_T} \times 100,$$

where

C'_T = the expected total cost of the optimal sampling plan when it is assumed that inspection error is present.

For example, if the inspection process includes errors of ($e_1 = 0.10$, $e_2 = 0.15$) then C'_T = \$1,945.78.

It is also instructive to note the percentage error which results when the sampling plan ($n = 126$, $n = 6$) is used instead of the optimal sampling plan when inspection error is present. In this case the percentage error is defined by

$$\Delta_3 = \frac{(C'_{TE} - C'_T)}{C'_T} \times 100.$$

These results are listed in Table VI. For example, if the inspection process includes errors of ($e_1 = 0.10$, $e_2 = 0.15$) then by designing for inspection error we have reduced the total cost per lot by \$567.76.

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A GENERAL TREATMENT OF UPPER UNBOUNDED AND BOUNDED HITCHCOCK PROBLEMS

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ABSTRACT

This paper is designed to treat (a) the problem of the determination of the absolute minimum cost, with the associated assignments, when there is no limit, N, on the number of parcels available for shipment in a modified Hitchcock problem. This is accomplished with the use of a transformed cost matrix, C^*, to which the so-called transportation paradox does not apply. The general Hitchcock solution using C^* gives the cost T^*, which is the absolute minimum cost of the original problem, as well as sets of assignments which are readily transformed to give the general assignments of the original problem. The sum of these latter assignments gives the value of N_u, the unbounded N for minimum cost. In addition, this paper is designed to show (b) how the method of reduced matrices may be used, (c) how a particular Hitchcock solution can be used to determine a general solution so that one solution using C^* can provide the general answer, (d) how the results may be modified to apply to problems with fixed N, and hence (e) to determine the function of the decreasing T as N approaches N_u, and finally (f) to provide a treatment when the supplies at origin i and/or the demands at destination j, are bounded.

1. INTRODUCTION

As indicated in [1], the lower bounded and partial upper bounded distribution model is a simple modification of the classical Hitchcock model in which the amount shipped from origin i to destination j, $i\epsilon I$ (a set of the first m integers), $j\epsilon J$ (a set of the first n integers), is determined so as to minimize

$$T=\sum_{I,J} x_{ij}c_{ij}, \tag{1.1}$$

subject to the modified conditions.

$$\sum_{J} x_{ij} \geqslant a_i, \qquad \sum_{I} x_{ij} \geqslant b_j \tag{1.2}$$

and for fixed N (the number of parcels available for shipment)

$$\sum_{I} a_i \leqslant N, \qquad \sum_{J} b_j \leqslant N, \qquad \text{and} \sum_{I,J} x_{ij} \leqslant N. \tag{1.3}$$

Charnes, Glover, and Klingman [1, p. 278] have related the solution of this problem to that of an

"equivalent" bounded variable (Hitchcock) problem which has, generally, much larger values of $a_i(b_j)$ with an additional row, $m+1$, an additional column, $n+1$, and a new fixed N

$$N_2 = (m+n)N - \left(m\sum_I a_i + \overline{n-1}\sum_J b_j\right) \tag{1.4}$$

Although $N_2 = N$, when $\Sigma a_i = \Sigma b_j = N$, it is generally much larger than N when N is taken greater than larger $(\Sigma a_i, \Sigma b_j)$. The authors suggest a stepping stone algorithm for obtaining the solution of the "equivalent" problem from which the solution of the original problem is obtained by deleting the elements of the last row and of the last column of the solution of the transformed problem.

This approach does give a solution for specified N, although a stepping stone method, without extension, usually gives only one solution when there may be many, but it does not answer the pertinent question as to how large the N must be to obtain the absolute smallest cost. For this the authors suggested [1, pp. 279 and 280] that the problem can be solved for enough different values of N so that eventually the N producing the smallest possible T can be determined.

Szwarc [4] has presented a variation of this latter problem in which there is no bound on the supply at each origin, i, nor on the demands at each destination, j. A major concern in his approach is the so-called transportation paradox—an increase in shipment size may lead to smaller cost, which he related to various types of problems involving the specifications of the supplies and demands. He used a stepping stone method for establishing a particular solution which he called the optimal solution [1, pp. 195 and 198–200]. He illustrated his method by application to three examples.

2. THE MODEL

The chief objective of this paper is to provide a general answer to the problem of determining the smallest possible T, and the associated allocations, no matter how large an N is needed. This lower bounded and upper unbounded model differs from the Charnes, Glover, Klingman model in that there is no fixed N and hence no (1.3). It is like the model of Szwarc [4] in which the solution process is directed toward the determination of the allocation, x_{ij}, which gives the smallest possible T no matter what the sizes of the supplies and demands.

3. AN EQUIVALENT PROBLEM

As with the Charnes, Glover, and Klingman treatment, that given here utilizes a second problem, a transform of the original problem whose solution(s) can be transformed back to obtain the solution(s) of the original problem. However, although the N (say N^*) of the transformed problem is fixed, the treatment here is designed to produce the general allocations, the smallest possible T, and the associated N, say N_u, for the original problem and not just a particular answer for some fixed N. In order to accomplish this, the transformations are based on an analysis of the relations within the cost matrix itself and not, as in the Charnes, Glover, and Klingman approach, on transformations of the a_i, b_j, and specified N. The main property of the transformed problem used here is that the cost matrix $C = [c_{ij}]_{m\times n}$ is transformed to a new cost matrix C^* having elements which are not subject to the so-called transportation paradox [4], [1, p. 277].

In this treatment the very unusual case with some $c_{ij} < 0$ is excluded since under the model speci-

fied above, the solution element x_{ij} can be made as large as desired, with $x_{ij}c_{ij}$ then as small as desired, so T can be made smaller than any specified negative number. But cases with one or more $c_{\gamma\delta}=0$ need not be excluded since, no matter how large $x_{\gamma\delta}$, $x_{\gamma\delta}c_{\gamma\delta}=0$ so T is not decreased by large values of $x_{\gamma\delta}$, although N_u can be finitely large.

Once an optimal Hitchcock solution is established for a problem with all $c_{ij} \geq 0$, the only way that T can be made smaller with increasing allocations is to replace some $x_{\alpha\beta} > 0$ by assigning it simultaneously to other values of c_{ij} whose sum is less than $c_{\alpha\beta}$, without making the total allocation to the row (column) less than $a_\alpha(b_\beta)$. Hence one of these c_{ij} must be in row α and another in column β. When the sum of these elements is less than $c_{\alpha\beta}$, with $x_{\alpha\beta} > 0$, then the value of T is always decreased by this operation. The decrease is greatest when the smallest element in row α, $c_{\alpha j^*}$, and the smallest element in column β, $c_{i^*\beta}$, are used. We have then for such elements, $c_{\alpha\beta}$, no matter whether $x_{\alpha\beta} > 0$ or not, and irrespective of the values of $a_\alpha > 0$ and $b_\beta > 0$,

$$c_{\alpha\beta} > c_{\alpha j^*} + c_{i^*\beta}. \tag{3.1}$$

For such elements we define

$$^*c_{\alpha\beta} = c_{\alpha j^*} + c_{i^*\beta}. \tag{3.2}$$

and know that $c_{\alpha\beta} - {}^*c_{\alpha\beta}$ is as large as possible. In the following discussion we use α, β to represent any i, j pair for which (3.1), and hence (3.2), hold.

We observe that any $x_{\alpha\beta} > 0$ can be moved to positions indicated by αj^* and $i^*\beta$ with a maximum decrease in T of $x_{\alpha\beta}(c_{\alpha\beta} - {}^*c_{\alpha\beta})$, and an increase in the total number of parcels shipped of $x_{\alpha\beta}$, without any violation of (1.2). Then the total maximum decrease in T, when all the $x_{\alpha\beta} > 0$ are so moved, is $\sum_{\alpha,\beta} x_{\alpha\beta}(c_{\alpha\beta} - {}^*c_{\alpha\beta})$ and the increase in the number of parcels shipped is $\sum_{\alpha,\beta} x_{\alpha\beta}$.

The transformed problem can now be stated. The transformed matrix is C^*, which is formed by replacing every $c_{\alpha\beta}$ by ${}^*c_{\alpha\beta}$, the remaining c_{ij} terms being unchanged, thus eliminating from the transformed matrix all elements which satisfy (3.1). The so-called paradox does not apply to the transformed matrix, since there is no way that its T can be made smaller, with $c_{ij} \geq 0$, by adding additional units of allocation.

The values of a_i and b_j are taken as those of the original problem. A general Hitchcock solution (i.e., a general solution with nonnegative x_{ij} and fixed a_i and b_j) is obtained for this transformed problem with its general solution indicated by $\{y_{ij}\}$ and the minimum cost by T^*. The solution elements associated with α, β terms are indicated generally by $y_{\alpha\beta}$ or, when α and β are specific (numerical) values, by $y^*_{\alpha\beta}$. The fixed N of the transformed problem is denoted by N^*.

No reduction in T^* is possible either (a) by using (3.1), since all situations in which it applies are removed, or (b) by increasing the assignments at specific i, j since no negative contribution is possible with ${}^*c_{ij} \geq 0$.

In an earlier paper which treated overshipments for the Hitchcock problem [3, sect. 12], the matrix C^* was called the "overshipment cost matrix," since solutions obtained with its use provide overshipments for the Hitchcock problem when such overshipments make T smaller. Although not strictly

applicable here where "overshipments" are recognized in the model, it may be used in the sense that the solution may require more supplies than the a_i and/or more demands than the b_j.

The solution of the original problem is then found by using the $y_{\alpha\beta}$ values as additions to the $y_{\alpha j^*}$ and $y_{i^*\beta}$ and setting $x_{\alpha\beta}=0$ to obtain the values of $x_{\alpha j^*}$ and $x_{i^*\beta}$. By multiplying (3.2) by $y_{\alpha\beta}>0$, we see that this operation has no effect on T^*, which is the absolute smallest sum for unbounded N. The value of N_u is $N^*+\sum_{\alpha,\beta} y_{\alpha\beta}$ and also $N_u=\sum_I \sum_J x_{ij}$. The supplies for this optimal solution are $a_i^*=\sum_J x_{ij}$, while the demands are $b_j^*=\sum_I x_{ij}$.

With consideration of the so-called transportation paradox not necessary in working with C^*, we need to treat only the case of the Hitchcock problem when $\sum_I a_i=\sum_J b_j$ and the modified problem with $\sum_I a_i \neq \sum_J b_j$.

4. TREATMENT WHEN $\Sigma a_i=\Sigma b_j$.

In this common case, the N^* is taken to equal $\sum_I a_i=\sum_J b_j$ since a larger N^* never causes a decrease in T^*. The solution, preferably a general one, follows as indicated in the section above.

It is not necessary to check the general values of y_{ij} to determine that each is nonnegative, since the problem posed involves x_{ij} and these must satisfy (1.2). In many problems one or more values of y_{ij} may serve as parameters of general solutions. However, these parameters must be determined subject to the conditions.

(4.1) $$x_{ij}\geq 0, \qquad a_i^*-a_i\geq 0, \qquad b_j^*-b_j\geq 0,$$

where $$a_i^*=\sum_J x_{ij} \qquad \text{and } b_j^*=\sum_I x_{ij}.$$

This treatment is illustrated by application to a general parametric solution of example 1 of Szwarc [4] in Table 1. (The details of the solution using the method of reduced matrices are presented in sect.

TABLE 1. *General solution for example 1 of Szwarc* [4]

45	5	12	13	15	a_i^*	$a_i^*-a_i$
10	*25	$10^{12-y_{22}}$	2	*30	$12-y_{22}$	$2-y_{22}$
15	$5^{15-y_{22}}$	$15^{y_{22}}$	*20	*10	15	0
20	*100	*65	$0^{15-y_{22}}$	2^{15}	$30-y_{22}$	$10-y_{22}$
b_j^*	$15-y_{22}$	12	$15-y_{22}$	15	$57-2y_{22}$	
$b_j^*-b_j$	$10-y_{22}$	0	$*2-y_{22}$	0		$12-2y_{22}$

$0\leq y_{22}\leq 2$ $\qquad T^*=225$

7). The x_{ij} are placed as post superscripts to the c_{ij}, the a_i and b_j appear in the left and upper margins with the a_i^* and b_j^* at the right and bottom, and their sum, N_u, in the lower right corner. The minimum cost is $T^* = 225$ cost units. An additional lower row and right column shows the values of $b_j^* - b_j$ and $a_i^* - a_i$. Application of (4.1) shows that $0 \leq y_{22} \leq 2$.

A similar treatment giving the general solution of example 2 of Szwarc [4] is presented in Table 2. The value of T is $T^* = \Sigma x_{ij} c_{ij} = 409$ cost units. Application of (4.1) to $x_{34} \geq 0$ shows $0 \leq y_{24} \leq 8$.

TABLE 2. *General solution for example 2 of Szwarc* [4]

46	4	11	12	8	11	a_i^*	$a_i^* - a_i$
7	14	*15	6^{12}	*13	*14	12	5
18	16^{4}	$9^{14-y_{24}}$	*22	$13^{y_{24}}$	*16	18	0
6	8	5	*11	$4^{8-y_{24}}$	5^{11}	$19 - y_{24}$	$13 - y_{24}$
15	12	4^{15}	*18	*9	*10	15	0
b_j^*	4	$29 - y_{24}$	12	8	11	$64 - y_{24}$	
$b_j^* - b_j$	0	$18 - y_{24}$	0	0	0		$18 - y_{24}$

$0 \leq y_{24} \leq 8$ $T^* = 409$

5. SOLUTION WHEN $\Sigma\, a_i \leq \Sigma\, b_j$

In this case there is no real loss in generality in assuming $\sum_I a_i < \sum_J b_j$, since the alternative situation can be reduced to this by interchanging the role of rows and columns. We take $N^* = \sum_J b_j$ since at least Σb_j parcels must be shipped and the adding of one or more new parcels can never decrease T^* when $c_{ij} \geq 0$. The real problem here is how to fill in the elements of an additional row, $m + 1$, to get the standard form of the Hitchcock problem with $\Sigma a_i = \Sigma b_j$. It is here proposed that $c_{m+1,\,j}$ be taken as the smallest element, c_{i^*j}, in column j to form the augmented C matrix. The augmented C^* matrix is then obtained from the augmented C matrix. Alternatively, it can be obtained from the C^* matrix by entering the c_{i^*j} in the $(m + 1)$st row since there is no α, β term in this row. The general solution yields T^* and the solution set $\{y_{ij}\}$.

(No larger element than c_{i^*j} in each column need be considered for row $(m + 1)$, for, if one such were selected under the conditions of the problem—no specifications on the $\Sigma b_j - \Sigma a_i$ additional values except that of smallest possible T—it could be replaced by c_{i^*j} to produce smaller T. It is possible that there may be more than one c_{i^*j} term having the same smallest column value in which case $y_{m+1,\,j}$ should be added to this set, forming alternate solutions. It should be pointed out, too, that this augmented C matrix can be used in finding the solution of the modified classical Hitchcock problem in which $\Sigma a_i < \Sigma b_j$ replaces $\Sigma a_i = \Sigma b_j$).

With the values of y_{ij} known, we first place all $y_{\alpha\beta} > 0$ terms in the two positions, which we may call concentration positions, and collect the results. Application of the solution process, thus far, for example 3 of Szwarc [4], is presented in Table 3A, and the results feature the parameter y_{52}. Then the values of $y_{m+1,\,j} > 0$ are added to the values of y_{i^*j} to obtain the x_{ij}. They are also added in the left

margin to obtain the values of a_i' whose sum is Σb_j. The values of a_i^* and b_j^* are computed as before and the values of $a_i^* - a_i'$ and $b_j^* - b_j$ to show where the $N_u - N^*$ units are located. The absolute smallest T is T^* of the augmented C^* solution and $N_u = b_j + \Sigma y_{\alpha\beta}$. These operations transform Table 3A to Table 3B where the unique (nonparameter) solution, x_{ij}, is indicated though the parameter y_{52} is featured in the a_i' and $a_i^* - a_i'$.

TABLE 3. *General solution for example 3 of Szwarc* [4]

TABLE 3A. *Solution for augmented* C

35	5	14	10	6	a_i^*	$a_i^* - a_i$
5	5	$8^{14-y_{52}}$	7	6	$14-y_{52}$	$9-y_{52}$
3	6	10	5	$5^{y_{52}-2}$	$y_{52}-2$	$y_{52}-5$
15	7^5	*15	3^{10}	*16	15	0
4	*15	*21	8^4	*18	4	0
8	5	$8^{y_{52}}$	3	$5^{8-y_{52}}$	8	0
b_j^*	5	14	14	6	39	
$b_j^* - b_j$	0	0	4	0		4

$5 \leq y_{52} \leq 8$ $T^* = 239$

TABLE 3B. *Solution for* C

35	5	14	10	6	a_i^*	$a_i^* - a_i'$
$5+y_{52}$	5	8^{14}	7	6	14	$9-y_{52}$
$11-y_{52}$	6	10	5	5^6	6	$y_{52}-5$
15	7^5	*15	3^{10}	*18	15	0
4	*15	*21	8^4	6	4	0
b_j^*	5	14	14	6	39	
$b_j^* - b_j$	0	0	4	0		4

$5 \leq y_{52} \leq 8$ $T^* = 239$

6. A PARTICULAR SOLUTION vs A GENERAL SOLUTION

In some situations it may be considered satisfactory to obtain a single (particular) solution even though it is recognized that there may be others. The treatment of this paper is available, with slight modification, for this. We simply obtain some solution using C^*, or augmented C^*, and transform it to obtain a particular solution of the original problem. It is recognized, of course, that the T^* obtained for a particular solution is the T^* of all solutions, so any solution determines the absolute smallest cost,

although the x_{ij} and the N_u may vary. However, one may not properly claim that such a particular solution is the solution unless he knows that there are no other solutions. An adequate treatment of the problem must provide all the answers and not just one of them.

General solutions permit the applications of additional conditions. For example, there may be an additional (fixed) cost associated with each shipment. In this case it is desired to minimize the number of shipments as well as the T. This is very easy if a general parametric solution is available. Some particular solution, if this property has not been used in its determination, is of little help. For, example, in the problem of Table 1, Szwarc [4] indicated that the solution is that given, in the present notation, by $y_{22} = 2$. This requires five shipments and is more than the four shipments of $y_{22} = 0$, easily obtained from the general result. In the problem of Table 2, Szwarc indicates that the solution is that given, in the present notation, by $y_{24} = 8$. This does have the smallest number of shipments, six, but so does the solution given by $y_{24} = 0$. His solution process does not generally produce such a solution or it would have done so in example 1. Since the solution of Table 3 is unique, it has the smallest number of shipments, but there appears to be nothing in the Szwarc solution process which leads one to know whether this solution has the smallest number of shipments nor even whether there are other solutions.

Again in problems in which different values of N_u give the same smallest T, it would seem desirable to ship the smallest number of parcels possible and so to determine what this number is. This is very easy when the general values, x_{ij}, are expressed in parametric form, since it is only necessary to assign values to the parameters so as to obtain the smallest $\Sigma\Sigma x_{ij}$. A particular solution, unless it is known to possess this property, is of little use. In the problem of Table 1, since $\Sigma\Sigma x_{ij} = 57 - 2y_{22}$, it is seen immediately that N_u is smallest, 53, when $y_{22} = 2$. In Table 2, with $\Sigma\Sigma x_{ij} = 64 - y_{24}$, we have the smallest N_u, 56, when $y_{24} = 8$. These correspond to the Szwarc answers, but there appears to be no consideration given in his statement of the transportation scheme to the question of the determination of smallest total shipments.

There may be other properties of a solution which are desired. These can commonly be determined from a general answer while a particular solution is of little help. While there are advantages in general results, it may be that a particular solution, with some methods, may be obtained much more easily. However, appropriate general methods are available for Hitchcock type problems, so that general solutions are obtained readily by hand or by computer [2 and its references] [3]. In any case a general solution should be available for such a useful problem [1]. Such a solution is outlined in the following section.

7. A GENERAL SOLUTION USING THE METHOD OF REDUCED MATRICES

The method of reduced matrices may be described as a method which transforms the problem (c_{ij} with a_i and b_j) to fit the solution as opposed to a stepping stone method which transforms the (feasible) solution to fit the problem. In the case of Hitchcock problems, the basic scheme of such a (matrix) transformation is very simple. We subtract constants from the elements of the rows (columns) of the cost matrix so that all transformed elements $c_{ij}^{(t)}$ are nonnegative until there are enough 0 elements so that all x_{ij} can be assigned to these. Since these subtractions from the rows (columns) have no effect on the values of the x_{ij}, although the value of T is decreased by the sum of the weighted values of the constants subtracted, it follows that there can be no smaller cost than that given by the solution, $\{x_{ij}\}$, since $\Sigma\Sigma x_{ij}c_{ij}^{(t)} = 0$.

The main problem is the determination of the constants to be subtracted. These subtraction operations are organized in four seccessive steps. The first step produces at least one 0 term in every row (column) so that the resultant matrix is "initially reduced." The next produces enough 0 terms in each row (column) when all 0 elements are weighted by their respective $b_j(a_i)$ to satisfy the $a_i(b_j)$ requirements, so that the matrix is "reduced grouped." The third step produces matrices from which integral x_{ij} can be determined using the 0 elements, although some x_{ij} may be negative. Such a matrix is said to be "algebraically reduced." The fourth step produces a matrix in which the x_{ij} associated with the 0 terms are all nonnegative. The resultant matrix is said to be "completely reduced" and a general solution can be obtained from the row and column specifications

$$\sum_I x_{ij} = b_j, \qquad \sum_J x_{ij} = a_i. \tag{7.1}$$

Procedures for these steps are described in Refs. [2 and references] and [3].

In steps 3 and 4, a formal means of finding the next transformation by computer (or by hand calculation) for two dimensional and multidimensional Hitchcock problems can be replaced, in the two dimensional case under consideration here, by an informal procedure which uses some x_{ij} as parameters [3] in satisfying the conditions (7.1).

The method of reduced matrices is directly applicable to the C^* and the augmented C^* matrices above. In order to make the presentation of the illustration compact, the net values of all the constants subtracted from the rows (columns) are indicated by the $u_i(v_j)$ in the bottom row and right marginal column of the appropriate matrices of Tables 4, 5, 6. The determination of these $u_i(v_j)$ is explained in the text below.

Some suggestions are made for the complete solution process which includes the determination of C^* (augmented C^*), the solution y_{ij} by the method of reduced matrices, and the computation of the x_{ij} from the y_{ij}.

(a) Write the cost matrix C, bordered above with the b_j and at the left with the a_i. Also write in parentheses in these borders the j^* for each i (i^* for each j), thus indicating the position of the smallest element in the row (column). Insert the values of c_{ij^*} in the column at the right and the values of c_{i^*j} in the row at the bottom. From this display one can see at a glance if $c_{ij} > c_{ij^*} + c_{i^*j}$. If so, use a star (asterisk) as an upper prescript for the c_{ij}. See Table 4A for an application to the illustration of Table 1.

(b) Construct an auxiliary table in which every starred term is replaced by the sum of the values at the bottom of its column and the right of its row to give ${}^*c_{\alpha\beta}$ of (3.2). The other c_{ij} terms are not changed nor are the top and left borders. The right and bottom borders are reserved for the values of u_i and v_j which, when subtracted from C^*, give $C^{*(t)}$. See Table 4B. The determination of the values of u_i and v_j, by the method of reduced matrices, is illustrated below.

(c) Determine a completely reduced matrix by subtracting constants indicated by the method of reduced matrices. Thus in Table 4, $v_2^{(0)} = 10$, $v_4^{(0)} = 2$, and $u_2^{(0)} = 5$ with a net reduction of $T^* = 10(12) + 2(15) + 5(15) = 225$ cost units. Here, as in many examples, the fairly trivial problems which are used to illustrate the details of alternative methods require only the first steps of the method of reduced matrices. The initially reduced matrix of Table 4C is completely reduced as may be seen from the fact that nonnegative values of y_{ij} can be assigned to the 0 terms. See Table 4A for an application to the illustration of Table 1.

TABLE 4. *Example 1 of Szwarc with reduced matrices*

TABLE 4A. C

45	5(2)	12(1)	13(3)	15(3)	c_{ij*}
10(3)	*25	10	2	*30	2
15(1)	5	15	*20	*10	5
20(3)	*100	*65	0	2	0
c^*_{ij}	5	10	0	2	

TABLE 4B. C*

45	5(2)	12(1)	13(3)	15(3)	u_i
10(3)	*7	10	2	*4	
15(1)	5	15	*5	*7	5
20(3)	*5	*10	0	2	
v_j		10		2	225

TABLE 4C. *Transformed* C* *and* y_{ij}

45	5(2)	12(1)	13(3)	15(3)	u_i
10(3)	*7	0^{10}	2	*2	
15(1)	0^5	$0^{y_{22}}$	$*0^{y_{23}}$	$*0^{y_{24}}$	5
20(3)	*5	$*0^{2-y_{22}}$	$0^{13-y_{23}}$	$0^{y_{34}}$	
v_j		10		2	225

where $y_{24} = 10 - y_{23} - y_{22}$ and $y_{34} = 5 + y_{22} + y_{23}$.

TABLE 5A. C

46	4(3)	11(4)	12(1)	8(3)	11(3)	c_{ij*}
7(3)	14	*15	6	*13	*14	6
18(2)	16	9	*22	13	*16	9
6(4)	8	5	*11	4	5	4
15(2)	12	4	*18	*9	*10	4
c_{i*j}	8	4	6	4	5	

TABLE 5B. C*

46	4(3)	11(4)	12(1)	8(3)	11(3)	u_i
7(3)	14	*10	6	*10	*11	0
18(2)	16	9	*15	13	*14	9
6(4)	8	5	*10	4	5	0
15(2)	12	4	*10	*8	*9	4
v_j	7	0	6	4	5	409

TABLE 5C. *Transformed* C* *and* y_{ij}

46	4(3)	11(4)	12(1)	8(3)	11(3)	u_i
7(3)	7	*10	0^{7}	*6	*6	0
18(2)	0^{4}	$0^{y_{22}}$	$*0^{y_{13}}$	$0^{y_{24}}$	$*0^{y_{25}}$	9
6(4)	1	5	*4	$0^{y_{34}}$	$0^{6-y_{34}}$	0
15(2)	1	$0^{11-y_{22}}$	$*0^{5-y_{23}}$	$*0^{y_{44}}$	$0^{y_{45}}$	4
v_j	7	0	6	4	5	409

where $y_{25}=14-y_{22}-y_{23}-y_{24}$, $y_{44}=8-y_{24}-y_{34}$ and $y_{45}=y_{22}+y_{23}+y_{24}+y_{34}-9$

TABLE 6A. *Augmented* C

35	5(1)	14(1)	10(3)	6(2)	c^*_{ij}
5(1)	5	8	7	6	5
3(3, 4)	6	10	5	5	5
15(3)	7	*15	3	*16	3
4(3)	*15	*21	8	*18	8
8	5	8	3	5	

TABLE 6B. *Augmented* C*

35	5(1)	14(1)	10(3)	6(2)	u_i
5(1)	5	8	7	6	0
3(3, 4)	6	10	5	5	0
15(3)	7	*11	3	*8	3
4(3)	*13	*16	8	*13	8
8	5	8	3	5	0
v_j	4	8	0	5	239

TABLE 6C. *Transformed augmented* C* *and* y_{ij}

35	5(1)	14(1)	10(3)	6(2)	u_i
5(1)	1	0^5	7	1	0
3(3, 4)	2	2	5	0^3	0
15(3)	0^5	$*0^{y_{32}}$	$0^{y_{33}}$	$*0^{y_{34}}$	3
4(3)	*1	$*0^{y_{42}}$	$0^{10-y_{33}}$	$*0^{y_{44}}$	8
8	1	$0^{y_{52}}$	3	$0^{8-y_{52}}$	0
v_j	4	8	0	5	239

where $y_{43} = 9 - y_{32} - y_{52}$, $y_{34} = 10 - y_{32} - y_{33}$, $y_{44} = y_{32} + y_{33} + y_{52} - 15$

(d) Assign the y_{ij} to the $c_{ij}^{*(1)} = 0$ terms. In Table 4C. y_{12} must be 10 and y_{21} must be 5. With arbitary y_{22} and y_{21} (or some other arbitrary y_{ij}) determine the additional y_{ij} to satisfy the row and column requirements. Insert the values of u_i and v_j in the right and lower borders and check the elements of C^* by adding the u_i and v_j.

(e) It is not necessary to determine what values of y_{ij} are admissable (nonnegative) since all that is needed is the admissibility of the x_{ij}. Add each $y_{\alpha\beta}^*$ to the x_{ij} in the row indicated by the parentheses above and to that in the column indicated in the parentheses at the left. These operations lead at once to the solution given in Table 1 where the smallest possible cost is $T^* = 225$ cost units and $N_u = 57 - 2y_{22}$ for $0 \leqslant y_{22} \leqslant 2$.

A parallel solution for the illustration of Table 2 is presented in Table 5. In this problem we go as far as step 2 to obtain the complete reduction indicated by the u_i, the v_j, and the T^* of Table 5B and Table 5C. The u_i and v_j are obtained, Step 1, by subtracting first $v^{(0)} = 8, 4, 6, 4, 5$ and $u^{(0)} = 0, 5, 0, 0$ to obtain the initially reduced matrix and then Step 2, by subtracting $v^{(1)} = -1, -4, 0, 0, 0$ and $u^{(1)} = 0, 4, 0, 4$ to obtain the reduced grouped matrix which is completely reduced. The double assignment of the $y_{\alpha\beta}^*$ terms leads to the general answer of Table 2.

The solution for the example of Table 3, where $\Sigma b_j - \Sigma a_i = 8$, is presented in Table 6. The matrix

C^* includes the c_{i*j} terms and the solution is carried out as above. Here $v^{(0)} = 5, 8, 3, 5$ and $u^{(0)} = 0, 0, 0, 5, 0$ is followed by $v^{(1)} = -1, 0, -3, 0$ and $u^{(1)} = 0, 0, 3, 3, 0$ with $T^* = 239$ cost units.

8. TRANSFORMATION OF A PARTICULAR SOLUTION TO A GENERAL SOLUTION

Many seem to prefer solution methods for Hitchcock problems which produce a particular solution. The result can be extended to produce general answers by adaptation to the final operations of the method of reduced matrices. Once a particular solution, $\{x_{ij}\}$, is known, one can find constants u_i and v_j where u_i is a constant subtracted from row i, and v_j is a constant subtracted from column j, such that

$$\left\{\begin{array}{ll} c_{ij} - u_i - v_j = 0 & \text{when } x_{ij} > 0 \\ c_{ij} - u_i - v_j \geq 0 & \text{otherwise} \end{array}\right\} \tag{8.1}$$

and, indeed, some of the methods provide such values. With these known, one forms the completely reduced matrix

$$c_{ij}^{(t)} = c_{ij} - u_i - v_j. \tag{8.2}$$

One continues by assigning the general values to the $c_{ij}^{(t)} = 0$ terms according to the method of reduced matrices.

In the case of the upper unbounded problem, the particular solution should be that for C^* so (8.1) and (8.2) apply to C^* and the allocations are the y_{ij}. In the illustrations of this paper, the technique leads to the matrices of Table 4C, Table 5C, and Table 6C, from which the general solutions of Tables 1, 2, 3 result.

9. TREATMENT FOR FIXED N AND INCREASING N

The basic problem of this paper, the determination of the absolute smallest cost with the associated allocations and N_u, when there is no limit on the number of parcels available for shipment, is treated extensively in the preceding sections. It is pointed out that there is no answer when the matrix has at least one negative c_{ij}, since the cost may be less than any negative number. When some $c_{ij} = 0$, though the value of N_u may be finitely large with smallest T, the smallest value of N_u is provided by this method. With these exceptions, the method provides general answers for all problems with finite a_i, b_j, and c_{ij}.

One may encounter problems, however, in which a completely general solution is too general to be useful. For example, if all c_{ij} are equal, say k, the set of all optimal solutions (no other solution has a smaller T) is the set of all feasible solutions and the formal general optimal solution is of little value.

In Hitchcock type problems, the generality of the solution usually increases with an increase in the number of equal terms in the rows (columns) of the cost matrix and/or the number of 0 terms of the completely reduced matrix. Thus, in the treatment of $\Sigma a_i < \Sigma b_j$, if there are two or more elements in column j having the same smallest value, not only should $y_{m+1,j} > 0$ be assigned to each of these in turn to lead to separate solutions which might be quite general, but it should be allocated to these in all combinatorial ways. Again it is possible that there might be two or more "equivalent" concen-

tration positions in the same row (column); i.e., positions having the same smallest row (or column) value with 0 as the completely reduced matrix transform, so that the $y_{\alpha\beta}$ terms should be distributed among these in all possible ways. This can be done, of course, and the techniques presented above apply to each such distribution. It seems probable that such an absolute general result would usually have academic, rather than practical, value as in the case $c_{ij}=k$ so that a particular solution, or some partially general solution, would be useful. However, it should be pointed out that practical, rather than academic, cost matrices usually do not have many equal c_{ij} in the row (column) so that this discussion here may be somewhat academic. With these theoretical limitations, and considering the fact that the techniques above reveal if, when, and how much complete generality is sacrificed, we can say that the methods enable us to determine the various allocations and N_u with smallest possible cost adequately and with relative ease.

Similar comments apply to the treatment to the case of fixed N which is, in a sense, an inverse treatment of that above based on the smaller values of $c_{\alpha\beta} - {}^*c_{\alpha\beta}$. In case many of these are equal, or have certain equal weighted sums, it may be also necessary to use a combinatorial approach as an aid in obtaining the formal solution for fixed N.

The model of Charnes, Glover, and Klingman (sect. 1) calls for fixed N. It is possible to use the results from C^* not only to provide the answer for this fixed N, but also to provide a formula (graph) showing the T values for all $N >$ larger $(\Sigma a_i, \Sigma b_j)$, a result which Charnes, Glover, and Klingman indicate [1, p. 279] would be very desirable.

It is implied in (1.3) and it is of course necessary that N be at least as large as larger $(\Sigma a_i, \Sigma b_j)$. The presentation above shows that N need not be greater than N_u. If, for some reason, it is necessary to ship all the supplies at the origins, with $N > N_u$, the proper way is to ship the N_u units optimally and then the $(N - N_u)$ remaining units according to the smallest c_{ij}, say c_{i*j*}. The optimal cost as a function of N, when $N > N_u$ and $c_{ij} \geq 0$, is then given by

$$T_N = T^* + (N - N_u)c_{i*j*}. \tag{9.1}$$

Modifications can be made if total shipments from i, or to j, are bounded.

However, the important problem is the determination of the optimal shipping cost, as a function of N, when

$$\text{larger } (\Sigma a_i, \quad \Sigma b_j) \leq N \leq N_u. \tag{9.2}$$

No matter whether $\Sigma a_i = \Sigma b_j$ or not, the lower limit of (9.2) relates to the solution of the Hitchcock problem and the upper limit relates to the upper unbounded problem of this paper. Of course, if there are no α,β terms with $x_{\alpha\beta} > 0$, the problems are one and the same.

When there are α,β terms with $x_{\alpha\beta} > 0$, the solution above is directed toward using (3.1) to the greatest extent possible to decrease T_N with increasing N. This is still the problem with fixed N, although the restriction on N means that we may not be able to use α, β terms to the fullest extent. Although we could start with $N =$ larger $(\Sigma a_i, \Sigma b_j)$ and increase it, see [4, 189], the approach here is to start with N_u and decrease it by replacing the unit in the αj^* position and the one in the $i^*\beta$ position by only one unit in the α,β position. This gives some values of $x_{\alpha\beta} > 0$, although they are all 0 in the optimal solution with unbounded N. We do this by using the smallest values of $c_{\alpha\beta} - {}^*c_{\alpha\beta}$ since it is the amount

by which T is increased as a result of "moving back" a pair of units, the reverse of the process by which we find N_u having the smallest T.

The first step in the process is the determination of any parameters of the general solution so as to obtain the smallest N_u for T^*. Thus in Table 1 we take $y_{22} = 2$ with $N_u = 53$ so we need to determine the value of T only for $45 < N < 53$. In Table 2, with $y_{28} = 8$, we need to determine optimal T for $46 < N < 56$. In Table 3C we need to determine optimal T for $35 < N < 38$.

The second step in the process is to write in a new table, see Table 7, each surplus $a_i^* - a_i$, $(b_j^* - b_j)$, in the right (bottom) margin since the move back operation cannot be applied to any element in a row (column) which does not have a surplus. These are readily available in the earlier results. Thus from Table 1, with $y_{22} = 2$, we have $a_3^* - a_3 = 8$ and $b_1^* - b_1 = 8$. From Table 2, with $y_{24} = 8$, we have $a_1^* - a_1 = 5$, $a_3^* - a_3 = 5$, and $b_2^* - b_2 = 10$. From Table 3 we have $a_1^* - a_1 = 9 - 5y_{52}$, $a_2^* - a_2 = y_{52} - 5$, and $b_3^* - b_3 = 4$. The new table also contains the values of $a_i(b_j)$ in the left and upper margins.

The third step consists in writing in the (numerical) values of every concentration element of the solution in every row (column) which has a surplus. (The other values of x_{ij} might be inserted also, but they are not included because no changes are made in them and the table should be constructed with only the necessary entries.) We also mark with an asterisk each α, β position which has made contribution to the concentration elements, and then the $x_{\alpha\beta}$ for these, indicating the numbers that are to be moved back. Then we subtract from each concentration element the subset of all possible $x_{\alpha\beta}$. The result shows the allocations except that the x_{ij} not associated with an α, β term nor a concentration element are not recorded as they are unchanged by these move back operations, for any (every) specific assignment to these $x_{\alpha\beta}$ terms. See Table 7A, 7B, 7C.

The fourth step consists in writing the value $c_{\alpha\beta} - {}^*c_{\alpha\beta}$ for each α, β value.

The fifth step consists in assigning successive integer values, consistent with every surplus, to the smallest values of $c_{\alpha\beta} - {}^*c_{\alpha\beta}$ to obtain the smallest

$$T = T^* + \Sigma\ (c_{\alpha\beta} - {}^*c_{\alpha\beta})x_{\alpha\beta}, \tag{9.3}$$

with a decrease in N of $\Sigma\ x_{\alpha\beta}$ where the summation holds for all $x_{\alpha\beta}$ moved back.

TABLE 7. *A general solution for general problems*

TABLE 7A. *From tables 1 and 4C*

45	5	12	13	15	$a_i^* - a_i$
10					
15	$13 - \left\{ \begin{matrix} x_{23} \\ x_{24} \end{matrix} \right\}$		${}^*x_{23}$ 15	${}^*x_{24}$ 3	
20			$13 - \{x_{23}\}$	$15 - \{x_{24}\}$	8
$b_j^* - b_j$	8				8

TABLE 7B. *From tables 2 and 5C*

46	4	11	12	8	11	$a_i^* - a_i$
7			$12 - \left\{ \begin{matrix} x_{23} \\ x_{43} \end{matrix} \right\}$			
18		$6 - \left\{ \begin{matrix} x_{23} \\ x_{25} \end{matrix} \right\}$	$^*x_{23}$ 7		$^*x_{25}$ 2	
6					$11 - \left\{ \begin{matrix} x_{25} \\ x_{45} \end{matrix} \right\}$	5
15		$15 - \left\{ \begin{matrix} x_{43} \\ x_{45} \end{matrix} \right\}$	$^*x_{43}$ 8		$^*x_{45}$ 1	
$b_j^* - b_j$		10				10

TABLE 7C. *From tables 3 and 6*

35	5	14	10	6	$a_i^* - a_i$
$5 + y_{52}$		$14 - \left\{ \begin{matrix} x_{32} \\ x_{42} \end{matrix} \right\}$			$9 - y_{52}$
$11 - y_{51}$				$6 - \left\{ \begin{matrix} x_{34} \\ x_{44} \end{matrix} \right\}$	$y_{52} - 5$
15		$^*x_{32}$ 4	$10 - \left\{ \begin{matrix} x_{32} \\ x_{34} \end{matrix} \right\}$	$^*x_{34}$ 8	0
4		$^*x_{42}$ 5	$4 - \left\{ \begin{matrix} x_{42} \\ x_{44} \end{matrix} \right\}$	$^*x_{44}$ 5	0
$b_j^* - b_j$	0	0	4	0	4

The final step consists in writing $T = T_N$ as a function of N when the values of $x_{\alpha\beta}$ as functions of N are inserted.

The entries of Table 7 serve as illustrations. In Table 7A, the concentration elements are x_{21}, x_{33}, and x_{34} and the corresponding α, β elements are x_{23} and x_{24}. The smaller $c_{\alpha\beta} - {}^*c_{\alpha\beta}$ is 3 for $\alpha = 2, \beta = 4$, so we assign successive positive integers to x_{24} as far as we can. Now from column 1 and row 3, with $x_{23} = 0$, we see that x_{24} can be as large as 8. For these values of x_{24} the increase in T is $3x_{24}$, so we have

$$T = (T^* = 225) + 3x_{24} \qquad \text{for } N = 53 - x_{24},\ 0 \leq x_{24} \leq 8 \tag{9.4}$$

and as a function of N

$$T_N = T^* + 3(53 - N) = 384 - 3N, \qquad 45 \leq N \leq 53. \tag{9.5}$$

This gives the optimal T for integral N from $N=45$ (Hitchcock) to $N=53$ (smallest T for any N) as required by (9.2).

It is thus seen that the graph of optimal T as a function of increasing N consists of the points corresponding to integral N on a straight line segment with slope -3. The corresponding x_{ij} for each N are obtained by placing $x_{23}=0, x_{24}=53-N$ in the entries of Table 7A. Other values of x_{ij} not appearing in Table 7A are those of Table 1 with $y_{22}=2$. These are $x_{12}=10$ and $x_{22}=2$.

The treatment of the second example is shown in Table 7B. The concentration values are x_{13}, x_{22}, x_{35}, and x_{42}. The corresponding $x_{\alpha\beta}$ are x_{23}, x_{25}, x_{43}, and x_{45}. The smallest admissible $c_{\alpha\beta}-{}^{*}c_{\alpha\beta}$ is 1 for $\alpha=4$, $\beta=5$. Since row 3 shows $x_{25}+x_{45}$ cannot be greater than 5, we can take $x_{25}=0$ and x_{45} any integer up to 5. When $x_{45}=5$, there is an adjustment surplus of five in column 2 and in row 1. Either x_{23} or x_{43} may then be used, but $c_{\alpha\beta}-{}^{*}c_{\alpha\beta}$ is smaller, seven, when $\alpha=2$, $\beta=3$. The full five units are available for x_{23}. Hence, we can write

$$T=(T^{*}=409)+(1)x_{45}]^{5}_{(1)}+(7)x_{23}]^{5}_{(2)}, \tag{9.6}$$

where $x_{45}]^{5}_{(1)}$ indicates that up to five units of x_{45} can be assigned first and $x_{23}]^{5}_{(2)}$ indicates that after the full five units are assigned, the x_{23} are assigned as needed to make specific N. When $x_{45}=5$ and $x_{23}=5$, all concentrations have been moved back and we have the Hitchcock T,

$$T_h=409+1(5)+7(5)=449 \text{ cost units}$$

as may be checked by the formal Hitchcock process.

Writing T as a function of N, we have the two intersecting line segments

$$\begin{aligned} T_N&=409+(56-N)=465-N \qquad 51\leqslant N\leqslant 56 \\ T_N&=414+7(51-N)=771-7N \qquad 46\leqslant N\leqslant 51. \end{aligned} \tag{9.7}$$

The corresponding x_{ij} are obtained by placing $x_{25}=0$, $x_{43}=0$, $x_{23}=0$, $x_{45}=56-N$ for $51\leqslant N\leqslant 56$ and by $x_{25}=0$, $x_{43}=0$, $x_{45}=5$, $x_{23}=51-N$ for $46\leqslant N\leqslant 51$. The other $x_{ij}>0$ values are those of Table 2 with $y_{24}=8$, i.e., $x_{21}=4$ and $x_{24}=8$.

The two formulae of (9.7) may be combined in the single formula

$$T_N=409+(56-N)+7(51-N) \qquad 46\leqslant N\leqslant 56 \tag{9.7a}$$

with the understanding that all negative values of $51-N$ are replaced by 0 and values of $56-N>5$ by 5.

A corresponding treatment of the entries of Table 7C, where the smallest $c_{\alpha\beta}-{}^{*}c_{\alpha\beta}$ is 4, when $\alpha=3$, $\beta=2$, shows a move back of four surplus units to x_{32}, since y_{52} may be taken as 5. We get

$$T_N=(T^{*}=239)+4(39-N)=395-4N \qquad 35\leqslant N\leqslant 39. \tag{9.8}$$

If, however, $c_{34}-{}^{*}c_{34}$ were three instead of eight, (c_{34} were 11 instead of 16), x_{34} could be no larger than three as y_{52} cannot be larger than eight. It is also necessary to move back one unit of x_{32} to obtain the result when $N=35$.

The method above is especially suited to common situations with a relatively small number of concentration elements and not too many rows (columns) with a surplus. The following more formal approach transforms the problem to a simple linear programming problem which is solved easily by substitution of side conditions in the objective function, and whose result can be ordered to give the functions of T_N.

We wish to find $x_{\alpha\beta} > 0$ which minimizes

$$T - T^* = \sum_{\alpha,\beta} (c_{\alpha\beta} - {}^*c_{\alpha\beta})\, x_{\alpha\beta}, \tag{9.9}$$

subject to

$$\sum_{\beta} x_{\alpha\beta} = f_\alpha(N) \leqslant a^*_\alpha - a_\alpha > 0$$

$$\sum_{\alpha} x_{\alpha\beta} = f_\beta(N) \leqslant b^*_\beta - b_\beta > 0, \tag{9.10}$$

where $f_\alpha(N)$ and $f_\beta(N)$ are of the form $k-N$. For the Hitchcock solution, (9.10) becomes

$$\sum_{\beta} x_{\alpha\beta} = a^*_\alpha - a_\alpha > 0, \qquad \sum_{\alpha} x_{\alpha\beta} = b^*_\beta - b_\beta > 0 \tag{9.10a}$$

since all α, β terms are moved back.

We expand (9.9) in terms of increasing values of $c_{\alpha\beta} - {}^*c_{\alpha\beta}$ and substitute corresponding values of (9.10) as needed to obtain the minimum sum, T_N, and the minimizing $x_{\alpha\beta} > 0$. Alternatively, we determine the $x_{\alpha\beta}$ terms for the Hitchcock solution using (9.10a) and determine the value for intermediate N by ordering the $x_{\alpha\beta}$ of the Hitchcock solution by increasing values of $c_{\alpha\beta} - {}^*c_{\alpha\beta}$.

As applied to the example of Table 7A, we wish to determine x_{23} and x_{24} to minimize

$$T = (T^* = 225) + 3x_{24} + 15x_{23} \tag{9.11}$$

when (column 1 and row 3)

$$x_{24} + x_{23} = 53 - N \leqslant 8 \qquad 45 \leqslant N \leqslant 53. \tag{9.12}$$

Substituting (9.12) in (9.10), we get

$$T = T^* + 3(53 - N) + 12x_{23},$$

which is the smallest when $x_{23} = 0$, so

$$T_N = T^* + 159 - 3N \qquad 45 \leqslant N \leqslant 53$$

s the T_N function as is given by (9.5).

As applied to the example of Table 7B, we wish to determine x_{23}, x_{25}, x_{43}, x_{45} to minimize

(9.13) $$T=(T^*=409)+x_{45}+2x_{25}+7x_{23}+8x_{43}$$

when

(9.14) $$x_{45}+x_{25}=56-N\leq 5 \qquad 51\leq N\leq 56$$

$$x_{23}+x_{43}=51-N\leq 5 \qquad 46\leq N\leq 51$$

$$x_{23}+x_{25}+x_{43}+x_{45}=56-N\leq 10 \qquad 46\leq N\leq 56.$$

By substituting the first two equations of (9.14) in (9.13) we get, with minimum maintained when $x_{25}=0$, $x_{43}=0$

(9.15) $$T_N=T^*+(56-N)+7(51-N) \qquad 46\leq N\leq 56$$

which, with the specification that $51-N$ in the formula is 0 when $N<51$, and 5 when $56-N>5$, satisfies the third equation of (9.14) and agrees with formula (9.7a).

In Table 7C, using (9.10a) we minimize

(9.16) $$T=T^*+4x_{32}+5x_{42}+x_{44}+8x_{34}$$

subject to the conditions,

$$\left.\begin{aligned} x_{32}+x_{42}&=9-y_{52}\\ x_{34}+x_{44}&=y_{52}-5\\ x_{32}+x_{34}+x_{22}+x_{44}&=4 \end{aligned}\right\} 5\leq y_{52}\leq 9.$$

Then

$$T=T^*+4(4)+1(y_{52}-5)+x_{42}+3x_{34}$$

which is the smallest T, when $y_{52}=5$, $x_{42}=0$, $x_{34}=0$, and hence $x_{44}=0$ so

$$T_h=T^*+4(x_{32}=4).$$

Then

(9.17) $$T_N=T^*+4x_{32} \qquad 0\leq x_{32}\leq 4$$

$$T^*+4(39-N) \qquad 35\leq N\leq 39.$$

In case there are several equal values of $c_{\alpha\beta}-{}^*c_{\alpha\beta}$, the linear programming problem may have many solutions, all of which should be examined. For example, if each of the four values of $c_{\alpha\beta}-{}^*c_{\alpha\beta}$ in Table 4B were unity, the function (9.13) becomes

(9.13a) $$T=T^*+x_{45}+x_{25}+x_{23}+x_{43},$$

so, using (9.10a), we have

(9.14a)
$$x_{25}+x_{45}=5$$

$$x_{23}+x_{43}=5$$

$$x_{23}+x_{25}+x_{43}+x_{45}=10.$$

Then

$$T_h=T^*+10,$$

and the solutions, $\{x_{ij}\}$, consist of all integers satisfying (9.14a). Since all $c_{\alpha\beta}-{}^*c_{\alpha\beta}$ are unity, we get

$$T_N=T^*+1(56-N) \qquad 46\leqslant N\leqslant 56,$$

no matter what the values of x_{ij}.

10. TREATMENT FOR BOUNDED SUPPLIES AND/OR DEMANDS

The total supplies, N, must be shipped through specific origins and are usually located there. In this case it seems more realistic to pose the problem in the form that the supplies at each origin (parcels acceptable at each destination) are specified rather than the total number of parcels, N. Using the notation of Szwarc [4] we take $\bar{a}_i \geqslant a_i$ to indicate the supplies at i and $\bar{b}_j \geqslant b_j$ to indicate the largest shipment acceptable at j. Once the values of $a_i^*(b_j^*)$ are calculated as indicated above, it is relatively easy to compare these values with $\bar{a}_i(\bar{b}_j)$. If $\bar{a}_i \geqslant a_i^*$ and $\bar{b}_j \geqslant b_j^*$, for all values of i and j, the completely optimal solution holds. If, however, some $\bar{a}_i < a_i^*$ and/or $\bar{b}_j < b_j^*$ some units of concentration elements must be moved back so as to satisfy the specifications. All that is needed are those values of $c_{\alpha\beta}-{}^*c_{\alpha\beta}$ used in determining T_N, which have made contribution to concentration elements. However, in many cases there are some $\bar{a}_i > a_i^*$ and/or some $\bar{b}_j > b_j^*$, so that the α, β terms and concentration values are but a subset of those used in determining T_N. For example in Table 7A with $b_1=10$, since $b_1^*=13$, we move back three units from x_{21} by taking $x_{23}=0$ and $x_{24}=3$. In Table 7B, if $a_1=10$ and $a_3=10$, since $a_1^*=12$ and $a_3^*=11$, we move back two units from x_{13} and one unit from x_{35} so $x_{23}=2$ and $x_{45}=1$. This move back may also lead to the satisfaction of some $b_j^*-\bar{b}_j>0$. In this illustration $b_2=18$, though less than $b_2^*=21$, is also satisfied when $x_{23}=2$ and $x_{45}=1$, since $6-x_{23}+15-x_{45}=18$. However, with $b_2=15$, it is necessary to move back three more units so we increase x_{45} by 3. Thus the solution for $a_1=10$, $a_3=10$, and $b_2=15$ is obtained by substituting $x_{45}=4$, $x_{23}=2$, $x_{43}=0$, $x_{25}=0$ in Table 7B.

The use of an additional column at the right with values of $a_i^*-\bar{a}_i>0$, and one at the bottom with values of $b_j^*-\bar{b}_j>0$ is suggested for tables like Table 7 with many such values.

Generally and more formally, one can use the simple linear programming technique in which the $x_{\alpha\beta}$ minimize

(10.1)
$$T=T^*+\Sigma x_{\alpha\beta}(c_{\alpha\beta}-{}^*c_{\alpha\beta})$$

conditioned by

$$\Sigma x_{\alpha\beta}]_i = a_i^* - \bar{a}_i \tag{10.2}$$

$$\Sigma x_{\alpha\beta}]_j = b_j^* - \bar{b}_j,$$

where $\Sigma x_{\alpha\beta}]_i$ indicates the sum of all $x_{\alpha\beta}$ which may be moved back from concentration elements in the row to α, β elements in that row. $\Sigma x_{\alpha\beta}]_j$ is defined similarly for column j. The summation of (10.1) applies to all these α, β terms.

The technique is similar to that of the section above and, in a sense, simpler, especially when $\Sigma(a_i^* - \bar{a}_1) = \Sigma(b_j^* - \bar{b}_j)$. However, modifications are in order when this is not true, as in the illustration above where $\Sigma(a_i^* - \bar{a}_i) = 2+1=3$, while $\Sigma(b_j^* - b_j) = 6$, since any move back operation changes the column frequency and the row frequency by the same amount. If $\Sigma(a_i^* - \bar{a}_i) < \Sigma(b_j^* - \bar{b}_j)$, it is suggested that, generally, the $x_{\alpha\beta}$ values of $(a_i^* - \bar{a}_i)$ be used primarily in minimizing the summation of (10.1) after which additional values of $x_{\alpha\beta}$ are determined for any adjusted $(b_j^* - \bar{b}_j)$ which are still not satisfied, as in the illustration above.

11. SUMMARY

The solution of the primary problem of this paper, the determination of the allocations to minimize the total shipment cost of a Hitchcock type problem with no limit on the number of parcels available for shipment, is obtained from the solution of that for a transformed cost matrix to which the transformation paradox does not apply. Methods which give general answers are indicated and the results provide immediate answers to such problems as the determination of those allocations (with absolute minimum cost) which permit, a) minimum number of parcels shipped, or b) minimum number of shipments.

It is possible to arrange the elements of the solution with certain $x_{\alpha\beta}$ terms variable to give general solutions for a general problem which includes more specific problems ranging from the Hitchcock problem (with supplies $= a_i$ and demands $= b_j$) to the primary problem above, as illustrated in Table 7. From this it is possible to obtain answers to such secondary problems as a) the determination of the allocations (and minimum cost) for a fixed total shipment (N), b) the determination of the minimum cost as a function of N, and c) the problem when some or all of the supplies and demands have upper bounds. These secondary problems are treated less formally and in less detail.

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LOCAL OPTIMIZATION FOR NONCONVEX QUADRATIC PROGRAMMING*

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1. INTRODUCTION

In this paper we present an algorithm for determining a local minimum to the following general quadratic programming problem

1) $$\text{Minimize } q(x) = p^T x + 1/2 x^T Q x$$

$$\text{subject to } Ax \geq b$$

$$x \geq 0,$$

where

Q is an n by n symmetric matrix
A is an m by n matrix
p is an n by 1 vector
b is an m by 1 vector.

We place no additional restrictions on the matrix Q. And so, the above problem is the minimization of a possibly nonconvex quadratic objective function subject to a set of linear inequality constraints.

The local minimum algorithm is based on work by Ritter [5, 6], Cottle and Mylander [1], Majthay [3], and Houthakker [2]. Our discussion begins with a necessary amount of theoretical background concerning the local minimum problem in section 2.

In section 3, we present the local minimum algorithm and proof of convergence.

The algorithm has been programmed in FORTRAN on the CDC 6500 system at Purdue University, and in section 4 some initial computational results are presented as well as some general remarks concerning the algorithm.

2. THEORETICAL FOUNDATIONS

In this section we are going to set forth the theory of an algorithm designed to find a local minimum to the general quadratic programming problem. In order to characterize a local minimum point we state a theorem proved in [3].

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THEOREM: A point x^* is a local minimum point to the general quadratic programming problem (1), if and only if there exist vectors y^*, u^*, v^* such that the Kuhn-Tucker conditions

$$u^* = p + Qx^* - A^T y^* \tag{2a}$$

$$v^* = -b + Ax^* \tag{2b}$$

$$0 = u^{*T}x^* + y^{*T}v^* \tag{2c}$$

$$0 \leqq u^*, v^*, x^*, y^* \tag{2d}$$

hold and such that for any nonzero vector w where

$$A_i w = 0 \qquad \text{if } y_i^* > 0 \tag{3a}$$

$$A_i w \geqq 0 \qquad \text{if } y_i^* = v_i^* = 0 \tag{3b}$$

$$w_j = 0 \qquad \text{if } u_j^* \geqslant 0 \tag{3c}$$

$$w_j \geqq 0 \qquad \text{if } u_j^* = x_j^* = 0 \tag{3d}$$

it follows that

$$w^T Q w \geqq 0 \tag{4}$$

where

$$A^T = [A_1^T, A_2^T, \ldots, A_m^T].$$

We shall refer to conditions (3) and (4) as the second order conditions. It is evident that the assumption of nondegeneracy precludes the possibility of (3b) and (3d), therefore the second order conditions become just (3a), (3c), and (4). We shall use this observation extensively. Finally we observe that the slack variables together with x are the primal variables, y and u are the dual variables.

In the case where the criterion function is convex the second order conditions are automatically satisfied when a solution to the Kuhn-Tucker conditions is obtained. Without the convexity assumption solutions to conditions in (2) may not satisfy (3) and (4). In order to develop a pivoting procedure that will eventually lead to a solution of (2), (3), and (4) (if a solution exists) we must obtain a characterization of (3) and (4). To do this we will present various theorems that develop the properties of matrices undergoing special pivot operations. These results are closely related to work done by others especially [5] and [1]. The first main result gives a characterization of the set of vectors w that satisfy (3) and (4) in terms of a matrix which gives a basis for the space. The matrix can be obtained by the pivoting process and condition (4) can be assured by the proper selection of pivots. The second main series of results

concerns the properties of matrices under a special class of pivots. These results together with the development of the generator for the space of the w vectors allows us to justify the algorithm given in section 3.

The optimality conditions set forth in Theorem 1 constitute an extremely demanding set of constraints to satisfy. If we want to satisfy them, and this is our final goal, we have to relax some of the constraints initially and satisfy only the rest. Conditions (3), (4) are the hardest to satisfy, therefore we shall want to satisfy them first, primarily at the expense of (2), especially (2d). Conditions (3), (4) can be satisfied in a manner referred to in the previous paragraph. After having found an initial solution that satisfies conditions (3), (4), and a part of condition (2), we want of course to find another one which, while still satisfying all the conditions the previous solution did, scores better at satisfying condition (2) or has a less stringent artificial restriction of the type described in the previous paragraph. Keeping this plan in mind the following questions must be answered first. "What can we assert of a solution that solves only a part of the system (2), (3), (4)? What more can possibly be said about the structure of an optimal solution? What sorts of operations can be carried out on a solution so as both to make some progress and to keep the desired properties of the present solution? The last question at once gives rise to a sequence of related questions: "How can we characterize solutions of (2) satisfying (3) and (4) in terms of the simplex tableau representing a solution, and what types of simplex tableau transformations will maintain these characteristics? The following sequence of statements is an attempt to give answers to these questions.

The following corollary to our basic theorem shows that if our starting point partially solves the Kuhn-Tucker conditions then the above mentioned restricted problem can be found by introducing very simple sorts of constraints.

THEOREM 2: Let us assume that x^*, y^*, u^*, v^*, solves the following incomplete system

$$\begin{aligned} u^* &= p + Qx^* - A^T y^*, \\ v^* &= -b + Ax^*. \\ 0 &= u_j^* x_j^* = y_i^* v_i^*, \qquad \forall_{i,j} \\ 0 &\leqq x^*, v^*. \end{aligned} \tag{5}$$

Then x^* is a local minimum point to the following restricted problem:

$$\min p^T x + \frac{1}{2} x^T Q x$$

$$\begin{aligned} A_i x &\geqq b_i \quad \text{if } v_i^* > 0 \text{ or } v_i^* = 0, \qquad y_i^* > 0 \\ A_i x &= b_i \quad \text{if } v_i^* = 0, \qquad y_i^* \leqq 0 \\ x_j &\geqq 0 \quad \text{if } x_j^* > 0 \text{ or } x_j^* = 0, \qquad u_j^* > 0 \\ x_j &= 0 \quad \text{if } x_j^* = 0, \qquad u_j^* \leqq 0, \end{aligned} \tag{6}$$

if and only if for any nonzero vector w where

$$(7) \qquad A_i w = 0 \quad \text{if } y_i^* > 0 \text{ or } v_i^* = 0, \qquad y_i^* \leqq 0$$

$$w_j = 0 \quad \text{if } u_j^* > 0 \text{ or } x_j^* = 0, \qquad u_j^* \leqq 0,$$

it follows that

$$(8) \qquad w^T Q w \geqq 0.$$

We can prove this theorem by applying the basic theorem to problem (6).

This result suggests that it might be worth while to relax the nonnegativity restrictions imposed on the dual variables at least temporarily, but to make sure the validity of conditions (7) and (8). This way we always have a local minimum point to some restricted problem and we only need to manage to be able to get eventually rid of the auxiliary restrictions.

For the sake of completeness we present the next theorem which gives the justification for limiting the search for a minimum point to basic complementary feasible solutions.

THEOREM 3: If we have a complementary feasible solution to the Kuhn-Tucker equations in which the column vectors corresponding to the positive variables are linearly dependent then we can construct another complementary feasible solution with fewer positive components and no higher objective function value assuming that the minimum exists.

PROOF: Assume that in a particular solution of the Kuhn-Tucker conditions (2) the positive variables are: u^1, y^1, x^2, v^2 and partition the matrices accordingly. The solution can be written:

$$(9) \qquad \begin{aligned} u^1 - Q_{12}x^2 + A_{11}^T y^1 &= p^1 \\ - Q_{22}x^2 + A_{12}^T y^1 &= p^2 \\ - A_{12}x^2 &= - b^1 \\ v^2 - A_{22}x^2 &= - b^2 \\ u^1, v^2, x^2, y^1 &> 0, \end{aligned}$$

as we can see the orthogonality conditions (2c) are satisfied.

Assume furthermore that the columns of (9) are linearly dependent:

$$(10) \qquad \begin{aligned} z^1 - Q_{12}z^3 + A_{11}^T z^4 &= 0 \\ - Q_{22}z^3 + A_{12}^T z^4 &= 0 \\ - A_{12}z^3 &= 0 \\ z^2 - A_{22}z^3 &= 0 \end{aligned}$$

From (9) and (10) we infer that for any real number r

$$(11) \qquad (u^1 + rz^1) - Q_{12}(x^2 + rz^3) + A_{12}^T (y^1 + tz^4) = p^1$$

$$-Q_{22}(x^2+rz^3)+A_{22}^T(y^1+tz^4)=p^2$$

$$-A_{12}(x^2+rz^3)=-b^1$$

$$(v^2+rz^2)-A_{22}(x^2+rz^3)=-b^2.$$

The corresponding objective value can be written as

12) $$\begin{aligned} q(x+rz) &= p^{2T}(x^2+rz^3)+1/2(x^2+rz^3)^TQ_{22}(x^2+rz^3) \\ &= p^{2T}(x^2+rz^3)+1/2(x^2+rz^3)^T(-p^2+A_{12}^T(y^1+rz^4)) \\ &= 1/2(p^{2T}x^2+b^{1T}y^1)+r/2(p^{2T}z^3+b^{1T}z^4) \\ &= q(x)+r/2(p^{2T}z^3+b^{1T}z^4). \end{aligned}$$

It is evident that the constraint

13) $$\begin{pmatrix} u^1 \\ v_2 \\ x^2 \\ y^1 \end{pmatrix} + r \begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{pmatrix} \geqq 0$$

llows v to assume both positive and negative values, and as $z \neq 0$ the feasible interval for r is not the hole line. At its finite boundary the vector in (13) has fewer positive components than it has for $r=0$. n the other hand we can see that if

$$p^{2T}z^3+b^{1T}z^4=0,$$

ıen the objective function does not depend on r while if

$$p^{2T}z^3+b^{1T}z^4\neq 0,$$

ıen by appropriate choice of the sign of r we are able to decrease the value of the objective function hile still maintaining feasibility. In the case when $z^i \geqq 0$, $i=1, 2, 3, 4$ and $p^{2T}z^3+b^{1T}z^4<0$, we can btain improvement in the objective function only through a positive r, which will never decrease the umbers of positive variables. This corresponds to an unbounded objective function, which is contrary o our assumption. This proves the theorem.

We now want to consider the properties of a quadratic tableau when certain pivots are allowed. The evelopment of these properties will allow us to characterize the space of w vectors and thus permit the ıe development of a scheme to satisfy the second order conditions. Assume now that we have a com-lementary basic solution to (2). Let u^1, y^1, x^2, v^2 be the basic variables. The value of the nonbasic ariables is zero by definition. The corresponding tableau can be obtained by pivoting on the marked lock in Figure 1.

(14)

		x^1	x^2	y^1	y^2
u^1	p^1	Q_{11}	Q_{12}	$-A_{11}^T$	$-A_{21}^T$
u^2	p^2	Q_{21}	Q_{22}	$-A_{12}^T$	$-A_{22}^T$
v^1	$-b^1$	A_{11}	A_{12}	0	0
v^2	$-b^2$	A_{21}	A_{22}	0	0

FIGURE 1. Set-up tableau.

THEOREM 4: In any tableau corresponding to a complementary basic solution of (2) the blocks defined by the basic dual-nonbasic primal and basic primal-nonbasic dual variables are symmetric while the block defined by the basic-nonbasic dual variables is the negative transpose of the block defined by the basic-nonbasic primal variables.

We can prove this lemma by actually carrying out the appropriate block-pivot, if we first show that the statement is true for the pivot-block. Except for the last statement, the proof is pure routine so we confine ourselves to the following

If we carry out the principal block pivot operation transforming the set up tableau into another tableau, we can see that the statement is true assuming it is true for the pivot block. For this reason we shall only have to prove it for the pivot block. This can be done by the following general result.

LEMMA: If the matrix

$$M=\begin{pmatrix} N & -P^T \\ P & 0 \end{pmatrix}$$

is invertible, and $N^T=N$, then

$$M^{-1}=\begin{pmatrix} R & -S^T \\ S & U \end{pmatrix},$$

where $R^T=R$, $U^T=U$.

PROOF: Denote I, E, F unit matrices of the appropriate size. From the definition of the inverse we have

(15)
$$\begin{pmatrix} N & -P^T \\ P & 0 \end{pmatrix} \quad \begin{pmatrix} R & W \\ S & U \end{pmatrix}=I$$

$$\begin{pmatrix} R & W \\ S & U \end{pmatrix} \quad \begin{pmatrix} N & -P^T \\ P & 0 \end{pmatrix}=I.$$

Transposing the latter equation, we get

(16)
$$\begin{pmatrix} N & P^T \\ -P & 0 \end{pmatrix} \quad \begin{pmatrix} R^T & S^T \\ W^T & U^T \end{pmatrix}=I.$$

On the other hand we can write (15) as

$$\begin{pmatrix} E & 0 \\ 0 & -F \end{pmatrix} \left[\begin{pmatrix} E & 0 \\ 0 & -F \end{pmatrix} \begin{pmatrix} N & -P^T \\ P & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & -F \end{pmatrix} \right]$$

$$\left[\begin{pmatrix} E & 0 \\ 0 & -F \end{pmatrix} \begin{pmatrix} R & W \\ S & U \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & -F \end{pmatrix} \right] \begin{pmatrix} E & 0 \\ 0 & -F \end{pmatrix} = I.$$

Pre- and postmultiplying this equation by $\begin{pmatrix} E & 0 \\ 0 & -F \end{pmatrix}$ we get

$$\begin{pmatrix} N & P^T \\ -P & 0 \end{pmatrix} \quad \begin{pmatrix} R & -W \\ -S & U \end{pmatrix} = I.$$

The uniqueness of the inverse yields by (16)

$$-W = S^T, \qquad R^T = R, \qquad U^T = U$$

hat was to be proven.

Our second theorem suggests that it might be very useful to characterize explicitly all the vectors satisfying

$$A_i w = 0 \qquad y_i^* > 0 \qquad \text{or } y_i^* \leqq 0, \qquad v_i^* = 0,$$

$$w_j = 0 \qquad u_j^* > 0 \qquad \text{or } u_j^* \leqq 0, \qquad x_j^* = 0.$$

As we only need to deal with basic complementary solutions and those can be symbolized by the partition (14) we can see that what we need is an explicit formula for the set

$$\{w_2 : A_{12} w_2 = 0\}.$$

As the answer seems to be of general interest we shall put it in a general way:

THEOREM 5: Let P be an r by s matrix of rank r and let N be an s by s symmetric matrix so that the matrix

$$M = \begin{pmatrix} N & -P^T \\ P & 0 \end{pmatrix}$$

is invertible. Denote

$$M^{-1} = \begin{pmatrix} N & -P^T \\ P & 0 \end{pmatrix}^{-1} = \begin{pmatrix} R & -S^T \\ S & U \end{pmatrix}$$

then the orthogonal complement of the rowspace of P is identical to the columnspace of R:

$$\{t : Pt = 0\} = \{t : t = Rs\},$$

where s is any vector with s components.

PROOF: From

$$\begin{pmatrix} N & -P^T \\ P & 0 \end{pmatrix} \quad \begin{pmatrix} R & -S^T \\ S & U \end{pmatrix} = I,$$

we get

$$PR = 0$$

(i.e., the column space of R is in the orthogonal complement of the row space of P). On the other hand the submatrix

$$(R \quad -S^T)$$

has rank s so the rank of R is at least $s - r$ what was to be proven.

COROLLARY: The set $\{w_2 : A_{12}w_2 = 0\}$ is equal to the column space of the matrix defined by the basic x and nonbasic u variables.

From this result we can immediately derive the following two results.

THEOREM 6: A nondegenerate complementary basic feasible solution to the Kuhn-Tucker Equations (2) is a local minimum point to the quadratic programming problem (1) if, and only if, the matrix defined by the basic x and nonbasic u variables in the corresponding tableau is positive semidefinite.

PROOF: Denote the inverse of the pivot block by

$$\begin{pmatrix} Q_{22} & -A_{12}^T \\ A_{12} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} R & -P^T \\ P & U \end{pmatrix}.$$

In effect we must prove that the matrix R must be positive semidefinite. From the above corollary and remembering the remark we made after our basic theorem on nondegeneracy we see that

$$W = \{w : w_1 = 0,\ A_{12}w_2 = 0\} = \{w : w_1 = 0,\ w_2 = Rt\}.$$

As a result for any w in W, we have

$$w^T Q w = w_2^T Q_{22} w_2 = t^T R Q_{22} R t,$$

where Q is partitioned as in Figure 1.

On the other hand from the inverse relation we have

$$Q_{22}R - A_{12}^T P = I.$$

Premultiplying by R and remembering that

$$A_{12}R = 0,$$

we get

$$RQ_{22}R = R,$$

i.e., for $w \epsilon W$

$$w^T Q w = t^T R t.$$

Here t is an unconstrained vector, so the last equality proves our theorem.

Our second result deals with the general case:

THEOREM 7: Let us assume that we have a complementary basic feasible solution to the incomplete Kuhn-Tucker system (5) and that the matrix in the corresponding tableau defined by the basic x and nonbasic u variables is positive semidefinite. Then the solution is a local minimum point to the following restricted problem

$$\min \; p^T x + 1/2 x^T Q x$$

(17)
$$A x \geqq b$$

$$A_i x = b_i \qquad \text{if } y_i \text{ basic and } y_i^* \leqq 0$$

$$x \geqq 0$$

$$x_j = 0 \qquad \text{if } u_j \text{ basic and } u_j^* \leqq 0.$$

PROOF: As we know from our basic theorem, the solution is optimal in problem (17) if and only if
(i) it solves the Kuhn-Tucker system of (17) that is (5), and
(ii) for any vector w satisfying

(18a) $$A_i w = 0 \qquad \text{if } y_i^* > 0 \qquad \text{or } i \text{ restricted}$$

(18b) $$A_i w \geqq 0 \qquad \text{if } y_i^* = v_i^* = 0 \qquad \text{and } i \text{ unrestricted}$$

(18c) $$w_j = 0 \qquad \text{if } u_j^* > 0 \qquad \text{or } j \text{ restricted}$$

(18d) $$w_j \geqq 0 \qquad \text{if } u_j^* = x_j^* = 0 \qquad \text{and } j \text{ unrestricted.}$$

We have

$$w^T Q w \geqq 0.$$

Our assumption and Theorem 2 yields that for any w in the set defined by (18a) and (18c) only, we have

$$w^T Q w \geqq 0$$

then, of course, this is true on the set (18) which is contained in it.

In the proof of this theorem we actually described the relevant necessary and sufficient conditions given in our theorem. The reason for this is that while it is very hard to verify the nonnegativity of a quadratic form on a cone, it is relatively easy to verify positive semidefiniteness. In fact, we shall be

able to assure it without even checking it. The following two statements deal with this possibility. Before stating them we introduce a term that will make it easier to express their essential content.

DEFINITION: For any complementary basic (not necessarily feasible) solution of the Kuhn-Tucker conditions (2), we shall refer to the matrix in the corresponding tableau defined by the basic x and nonbasic u variables as the *critical* matrix.

Using this term we can express the essential ideas of the following statements by saying that the positive semidefiniteness of the critical matrix is hereditary in certain sorts of pivot operations.

THEOREM 8: Assume that the critical matrix in the tableau corresponding to a complementary basic solution of the Kuhn-Tucker conditions is positive semidefinite or empty. Carry out a pivot transformation on this tableau using a positive diagonal element. The critical matrix in the resulting tableau is positive semidefinite.

PROOF: We can prove our theorem by actually carrying out the pivot transformation and using Theorem 4 which presents the basic symmetry relations in any tableau corresponding to a complementary basic solution. Four cases are to be tested.

(i) The pivot element p is in the current critical matrix.

We can assume without any loss of generality that it is the last element on the main diagonal, i.e., the form of the current, positive semidefinite, critical matrix is

$$\begin{pmatrix} M & m \\ m^t & p \end{pmatrix}.$$

After pivoting on the positive number p the size of the critical matrix decreases by one and the new critical matrix becomes

$$M - \frac{1}{p}\, mm^T.$$

We know that for any vector $\begin{pmatrix} z \\ w \end{pmatrix}$

$$0 \leqq (z^T w) \begin{pmatrix} M & m \\ m^T & p \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = z^T M z + 2m^T z w + p w^2,$$

i.e., this inequality is valid for

$$w = -\frac{1}{p}\, m^T z$$

$$0 \leqq z^T M z - \frac{1}{p}\,(m^T z)^2 = z^T (M - \frac{1}{p}\, mm^T) z$$

and this last inequality is true for any vector z.

(ii) Assume now that the pivot element is on the main diagonal of the block defined by the basic v and nonbasic y variables.

We observe first that the positive semi-definiteness of the critical matrix implies the positive semi-

definiteness of the matrix defined by the basic primal nonbasic dual variables which is either the zero matrix or has the form

$$\begin{pmatrix} M & MA_{22}^T \\ A_{22}M & A_{22}MA_{22}^T \end{pmatrix},$$

where M is the critical matrix defined by x_2 and u_2 while the rest is defined by v_2 and y_2.

If z and w are arbitrary vectors of the appropriate size we have

$$(z^T, w^T)\begin{pmatrix} M & MA_{22}^T \\ A_{22}M & A_{22}MA_{22}^T \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}=(z^T+w^TA_{22})M(z+A_{22}^Tw) \geqq 0.$$

After this observation, which seems to be of interest by itself, we can follow essentially the same line we followed in (i) as the new critical matrix will be the transform of M.

(iii) If the pivot element is in the block defined by the basic u nonbasic x variables then the critical matrix will increase. By using Theorem 4 we again know that the new critical matrix will be the transform of the current matrix

$$\begin{pmatrix} M & -m \\ m^T & p \end{pmatrix},$$

where M is the current critical matrix, and p is the positive pivot element. The new critical matrix will be

$$\begin{pmatrix} M+\frac{1}{p}\,mm^T & \frac{1}{p}\,m \\ \frac{1}{p}\,m^T & \frac{1}{p} \end{pmatrix}$$

For any vector z and number w we have

$$(z^T, w)\begin{pmatrix} M+\frac{1}{p}\,mm^T & \frac{1}{p}\,m \\ \frac{1}{p}\,m^T & \frac{1}{p} \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}= z^TMz+\frac{1}{p}\,(w+m^Tz)^2 \geqq 0.$$

(iv) Finally if the pivot element is in the block defined by the basic y nonbasic v variables, the size of the critical matrix does not change. By using Theorem 4 again, we can see easily that the new critical matrix is of the form

$$M+\frac{1}{p}\,mm^T,$$

what is certainly positive semidefinite if M, the old critical matrix, is.

THEOREM 9: Assume that the critical matrix in the tableau corresponding to a complementary basic solution of the Kuhn-Tucker conditions is empty or positive semidefinite. Carry out a two by two block-pivot transformation on this tableau in which the pivot block is defined by a basic primal variable, a basic dual variable and the corresponding (nonbasic) complementary variables assuming that

(i) the element defined by the chosen basic primal and its nonbasic complementary variable is zero, and

(ii) in case the critical matrix is not empty the defining basic dual variable is a y variable.

The critical matrix in the resulting tableau will be positive semidefinite.

PROOF: By referring to Theorem 4 and to regularity, we see that the structure of the two by two pivot block is as follows:

$$\begin{pmatrix} q & -p \\ p & 0 \end{pmatrix},$$

where q is a real number and p is different from zero. The inverse of this matrix is

$$\begin{pmatrix} 0 & \frac{1}{p} \\ -\frac{1}{p} & \frac{q}{p^2} \end{pmatrix}$$

There are three cases we have to check:

(i) In the initial tableau the critical matrix is empty. After carrying out the indicated pivot the resulting critical matrix has zero as its single element by (20).

(ii) The critical matrix is not empty and the defining basic variable is an x variable. As a result, the size of the critical matrix decreases by one. As the current critical matrix is positive semidefinite by assumption and as an element on its main diagonal, defined by the chosen basic x and its nonbasic complement, is zero, we know that the whole row and column defined by this diagonal element must be zero. By virtue of this and Theorem 4, we can depict the situation before pivoting as follows:

$$\left[\begin{array}{ccc} \begin{pmatrix} M & 0 \\ 0^T & 0 \end{pmatrix} & & \begin{matrix} -r \\ -p \end{matrix} \\ \begin{matrix} r^T & p \end{matrix} & & q \end{array}\right].$$

Here we indicated the current critical matrix by parentheses and the pivot block by brackets and omitted the part of the tableau unessential in our reasoning.

As we know the new critical matrix becomes

$$\hat{M} = M - (0 \quad -r) \begin{pmatrix} \frac{q}{p^2} & \frac{1}{p} \\ \frac{-1}{p} & 0 \end{pmatrix} \begin{pmatrix} 0^T \\ r^T \end{pmatrix} = M$$

which is a positive semidefinite matrix.

(iii) In the remaining case, the basic primal is v variable and the basic dual is y variable. The size of the critical matrix does not change. The situation can be appropriately depicted by the following tableau in which we again omit the unessential part and use the symmetry relations established in Theorem 4.

$$\begin{array}{|ccc|}\hline M & -r & 0 \\ r & \begin{bmatrix} q & -p \\ p & 0 \end{bmatrix} & \\ 0 & & \\ \hline \end{array}$$

Here M is the current critical matrix and we applied the byproduct of Theorem 8, which asserts that the positive semi-definiteness of the critical matrix implies the positive semi-definiteness of the matrix defined by the basic primal variables and their complement.

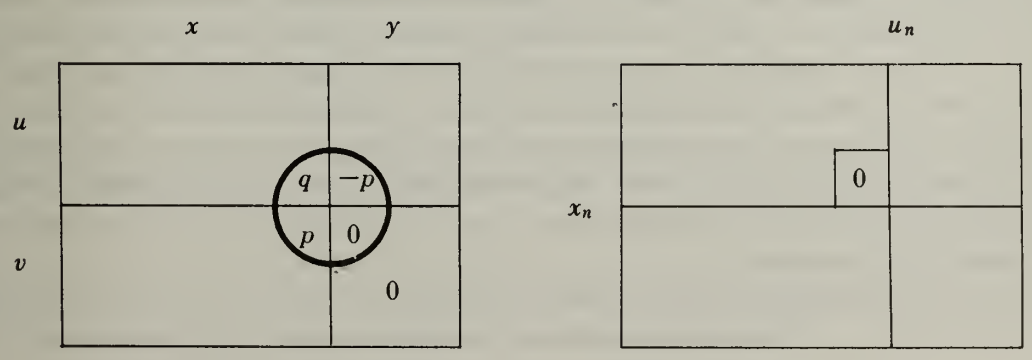
The new critical matrix becomes

$$\hat{M}=M-(-r \quad 0)\begin{pmatrix} 0 & \frac{1}{p} \\ \frac{-1}{p} & \frac{q}{p^2} \end{pmatrix}\begin{pmatrix} r^T \\ 0^T \end{pmatrix}=M,$$

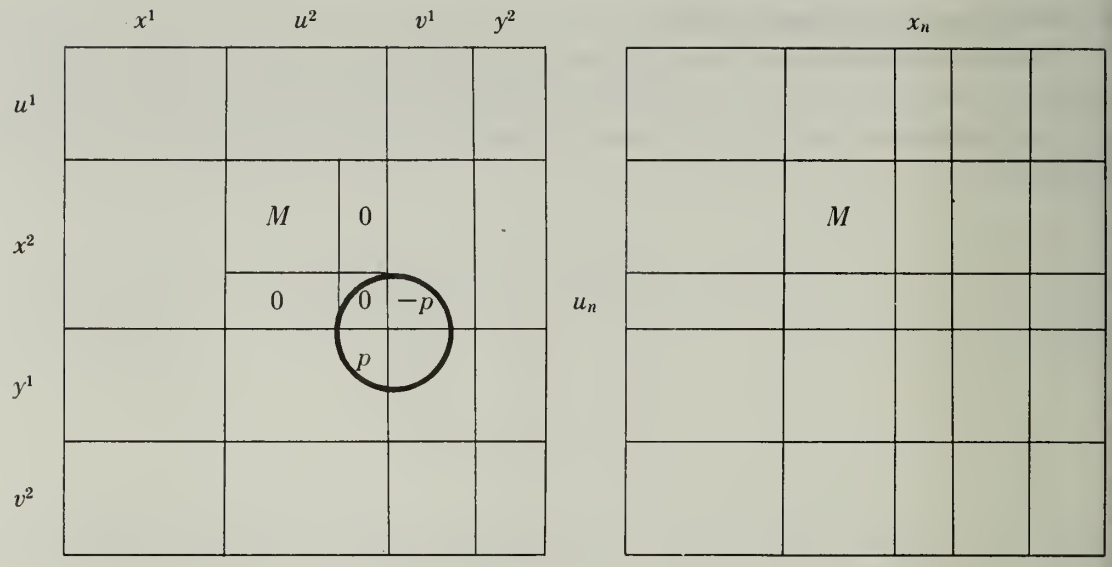
what proves the theorem.

The following figures might help in making both the statement and its proof clearer:

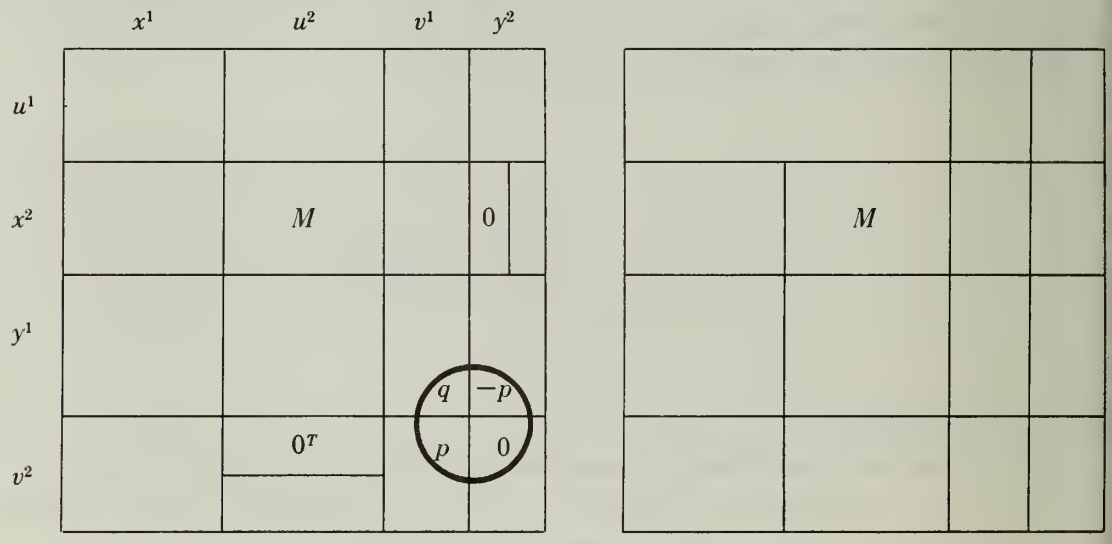
In the picture we marked the pivot block by a circle and the critical matrix by a thick square.



(ii)



(iii)



Let us return now to an earlier statement, to Theorem 7. It could be paraphrased as follows: any primal feasible basic solution of the Kuhn-Tucker equations is a local minimum point to a problem we obtain from our original problem by restricting certain nonbasic primal variables to the value zero.

As it is computationally inconvenient to introduce several additional restrictions at a time and as these new restrictions have a different form, being equalities, it seems to be more advantageous to introduce a single new constraint in inequality form that is equivalent to the set or a subset of the equality constraints. Such a constraint may be: the sum of the restricted variables should be nonpositive. This constraint and the nonnegativity ones together restrict the indicated variables to zero.

Some theoretical problem is, however, still unsolved: What is the transformed form of the Kuhn-Tucker equations of the restricted problem. The difficulty is caused by introducing a constraint that involves some of the v variables.

To be more specific assume that in the basic complementary solution to be restricted, the basic variables are u^1, x^2, y^1, v^2, i.e., we have additional restrictions on some of the components of x^1 and

v^1. The new constraint can be written as

$$c^T x^1 + d^T v^1 \leqq -b_{m+1} \tag{21}$$

where

$$c \geqq 0, \qquad d \geqq 0 \qquad \text{and } b_{m+1} = 0.$$

The constraints can be written as

$$v^1 = -b^1 + A_{11}x^1 + A_{12}x^2$$

$$v^2 = -b^2 + A_{21}x^1 + A_{22}x^2$$

$$d^T v^1 + v_{m+1} = -b_{m+1} + c^T x^1.$$

If we write the appropriate Kuhn-Tucker equations and carry out the necessary block pivot making x^2 and y^1 basic the last row will remain unchanged as its elements in the pivot column are zero. After that we only need to refer to Theorem 4, to see the validity of the following

THEOREM 10: The transformed tableau of the Kuhn-Tucker equations corresponding to the restricted problem in which we impose restriction (21) to the original constraints can be written as

	x^1	u^2	v^1	y^2	y_{m+1}
u^1					c
x^2					0
y^1					d
v^2					0
v_{m+1}	$-b_{m+1}$	$-c^T$	0	$-d^T$	0	0

In other words we can write the new constraint formally to the transformed tableau as if it were an original one and we would impose restrictions on the x variables.

We have denoted the right-hand side of the new constraint by $-b_{m+1}$ instead of by simply writing 0 because we want to relax this temporary constraint. We can do so by treating b_{m+1} as a parameter the value of which is originally zero and negative in general.

We would like to see how the value of our objective function depends on the value of this parameter. Before answering the question, however, we should state it more exactly.

Assume we have a complementary basic solution to the Kuhn-Tucker system of the problem with the additional constraint that it is primal feasible for some value of the parameter b_{m+1}. If we fix the basis and define the nonbasic variables to be zero as customary the solution defined by this basis will in general depend on b_{m+1}. Except for the case where we cannot change the value of b_{m+1} without losing feasibility we have a finite or infinite interval where the solution defined is feasible. Our question is: how will the value of the objective depend on b_{m+1} in this feasibility interval?

For any complementary solution of the Kuhn-Tucker equations we have

$$0 = u^T x + y^T v = p^T x + x^T Q x - b^T y,$$

which yields

$$p^T x + 1/2 x^T Q x = 1/2 (p^T x + b^T y). \tag{22}$$

Let us assume that in a complementary basic solution of the Kuhn-Tucker equation u^1, x^2, y^1, v^2 are the basic variables. Denote the basis by B. Define the values of the nonbasic variables to be zero, especially $x^1 = 0$, $y^2 = 0$, and treat the vectors p^2 and b^1 as variables while keeping B fixed. The pivot block is

$$\begin{bmatrix} Q_{22} & -A_{12}^T \\ A_{12} & 0 \end{bmatrix}.$$

Denote its inverse by

$$\begin{bmatrix} R & -S^T \\ S & U \end{bmatrix}$$

as above. We can express x^2, y^1 in terms of p and b from the Kuhn-Tucker equations as follows

$$\begin{bmatrix} x^2(p, b|B) \\ y^1(p, b|B) \end{bmatrix} = -\begin{bmatrix} R & -S^T \\ S & U \end{bmatrix} \begin{bmatrix} p^2 \\ -b^1 \end{bmatrix}$$

showing a linear relationship defined for any (p, b). We can write $x^1(p, b|B) \equiv -0$, $y^2(p, b|B) \equiv 0$.

Define the function $f(p, b)$ by

$$f(p, b|B) = 1/2(p^{2T} x^2(p, b) + b^{1T} y^1(p, b))$$

$$= -1/2(p^2, -b^1) \begin{pmatrix} R & -S^T \\ S & U \end{pmatrix} \begin{pmatrix} p^2 \\ -b^1 \end{pmatrix}$$

$f(p, b|B)$ is clearly defined on $R^n \times R^m$ and it is differentiable there. Its gradient can be expressed in the form:

$$\nabla f(p, b|B) = \begin{pmatrix} 0 \\ x^2(p, b|B) \\ y^1(p, b|B) \\ 0 \end{pmatrix} = \begin{pmatrix} x^1(p, b|B) \\ x^2(p, b|B) \\ y^1(p, b|B) \\ y^2(p, b|B) \end{pmatrix}$$

As in our introductory remark we have already pointed out we have.

$$f(p, b|B) = q(x(p, b|B)).$$

Thus we proved the following*:

THEOREM 11:

$$\nabla_{p,b}q(x(p, b|B)) = \begin{pmatrix} x(p, b|B) \\ \\ y(p, b|B) \end{pmatrix}$$

From this Theorem we can immediately see the validity of the following Corollary.

COROLLARY: If in any complementary basic solution the dual variable y_{m+1} corresponding to the auxiliary constraint is not in the basis then the value of the objective function does not depend on the parameter b_{m+1} while b_{m+1} is varying in its feasibility interval. If y_{m+1} is basic and positive then the objective is a strictly increasing function of b_{m+1}.

3. LOCAL MINIMUM ALGORITHM

The results of the previous section can be summarized as follows:

If we have a primal feasible basic complementary solution to the Kuhn-Tucker equations (2a) (2b) and the corresponding critical matrix is positive semidefinite then we can maintain both complementary and positive semidefiniteness of the critical matrix by carrying out a sequence of admissible pivot operations. The current basic solution can always be interpreted as a local minimum point to some restricted problem. Furthermore we know how the criterion function varies as we relax the temporary constraint. The theoretical development in section 2 clearly indicates the nature of the algorithm; however, we still have a certain freedom in constructing the actual algorithm and shall use it to improve computational efficiency. This observation will be employed in the choice of our temporary constraint.

As the formal description of the algorithm assumes an initial primal feasible complementary basic solution a few remarks are in order about initialization. It is easy to see that we can proceed as follows; carry out any sequence of desired pivot operations on matrix A making some x variables basic and some v variables nonbasic until we get primal feasibility and carry out the symmetric sequence of pivots on the skew-symmetric counterpart $-A^T$ thereby making the complements of the new basic primal variables nonbasic and the complements of the new nonbasic primal variables basic.

The last remark in the previous section assures us that we can interpret the resulting tableau as the Kuhn-Tucker tableau of a new, equivalent, quadratic programming problem and we can proceed by solving this new problem. For bookkeeping purposes, however, it seems to be more advantageous to regard the tableau as a transformed tableau of the original problem. By Theorem 9 we know that the pivot operations used in the initialization are admissible ones therefore the critical matrix is positive semidefinite. In fact it is easy to see that it is actually zero.

We shall now proceed to a discussion of the basic steps of the algorithm. The method utilized is similar to that presented by Cottle and Mylander [1] and Ritter [6], but it is also based on the original idea by Houthakker [2] further discussed by van de Panne and Whinston [7].

Before listing the steps of the algorithm we display the tableau and define the terms used:

$w_i \equiv$ variable in the basis and the current value of it $i = 0, 1, \ldots, n+m$

$z_i \equiv$ variable not in the basis $i = 0, 1, \ldots, n+m$. The pair of variables (w_i, z_i) form a complementary pair.

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*$q(x(p, b|B))$ is not necessarily a minimum of q on some set we are working with.

$m_{ij} \equiv$ the element in row i column j of the tableau.

$q_i \equiv$ constant term in the value of the basic variable w_i $i = 0, 1, \ldots, n+m$

$q_{\lambda i} \equiv$ coefficient of the linear term in the value of the basic variable w_i to be multiplied by the parameter λ.

Thus we have the following tableau, Figure 2.

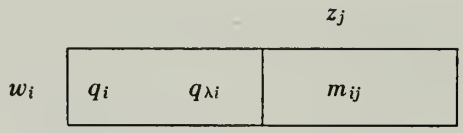

FIGURE 2. Kuhn-Tucker tableau.

Initially we have:

$$w_i = u_i = p_i \qquad i = 1, \ldots, n$$

$$w_i = v_{i-n} = -b_{i-n} \qquad i = n+1, \ldots, n+m$$

$w_0 \equiv$ slack variable for the auxiliary constraint.

Thus we have

$$\begin{aligned} q_0 &= 0 \\ q_i &= p_i \qquad i = 1, \ldots, n \\ q_i &= -b_{i-n} \qquad i = n+1, \ldots, n+m \\ q_{\lambda 0} &= 1 \\ q_{\lambda i} &= 0 \qquad i = 1, \ldots, n+m. \end{aligned}$$

Initially the tableau has the form in Figure 3

				$z_1 \ldots$		z_n	$z_{n+1} \ldots$	z_{n+m}	z_0
	Basic variable	Value 1	Value λ	x_1		x_n	y_1	y_m	y_{m+1}
w_1	u_1	p_1	0	q_{11}	$---$	q_{1n}	$-a_{11}--$	$-a_{m1}$	e_1
.	.			.		.	.	.	.
.	.			.		.	.	.	.
.	.			.		.	.	.	.
w_n	u_n	p_n	0	q_{n1}	$---$	q_{nn}	$-a_{1n}--$	$-a_{mn}$	e_n
w_{n+1}	v_1	$-b_1$	0	a_{11}	$---$	a_{1n}	$0--$	0	0
.	.	.		.		.	.	.	.
.	.	.		.		.	.	.	.
.	.	.		.		.	.	.	.
w_{n+m}	v_m	$-b_m$	0	a_{m1}	$---$	a_{mn}	$0--$	0	0
w_0	v_{m+1}	0	1	$-e_1$	$---$	$-e_n$	$0--$	0	0

FIGURE 3

In effect, we have introduced the more conventional notation

$$\lambda = -b_{m+1}$$

and

$$e_j = -a_{m+1,j} \qquad j = 1, \ldots, n,$$

where the values of the coefficients e_j will be specified later. We shall refer to the constraint corresponding to w_0 as the temporary auxiliary constraint or following Houthakker [2] as the capacity constraint.

For explanatory purposes we shall assume that the value of the basic variables is different from zero.

And now, we present the steps of the algorithm:

As a result of the initialization we have a primal feasible complementary basic solution with the corresponding tableau. Throughout the initialization we define

$$m_{0j} = 0 \qquad j = 1, \ldots, n+m$$

and

$$\lambda = 0.$$

STEP 1. Determine if all the values of the basic variables are greater than zero. If yes, stop. The present solution is an optimal solution. If not proceed to Step 2.

STEP 2. We have at least one basic variable that is negative. Define

$$J = \{j \neq 0 : w_j < 0, w_j \text{ is a dual variable}\}.$$

Define the last row of the tableau by introducing

$$m_{0j} = \begin{cases} 0 & j \notin J \\ -1 & j \in J \end{cases}.$$

We are relatively free in defining the numerical value of e_j, but it must have the following properties:

$$e_j \geqq 0, \qquad j \in J$$

$$\sum_{j \in J} e_j > 0,$$

and

$$w_j e_j < 0,$$

for at least one $j \in J$. The choice $e_j = 1$ for $j \in J$ can thus be taken. Proceed to Step 3.

STEP 3. Define r by

$$\frac{w_r}{e_r} = \min_{j \in J} \left\{ \frac{w_j}{e_j} : e_j > 0 \right\},$$

and carry out the double pivot operation

$$< w_r, z_0 >, \qquad < w_0, z_r >.$$

Proceed to step 4.

STEP 4. Case (i) If $q_{\lambda j} \geqq 0$, $j=0, 1, \ldots, n+m$ stop. The objective function is not bounded from below.

Case (ii) There is at least one j for which $q_{\lambda j} < 0$. Determine r by

$$\frac{q_r}{q_{\lambda r}} = \operatorname*{Min}_{j \in J} \left\{ \frac{q_j}{q_{\lambda j}} : q_{\lambda j} < 0 \right\}$$

If $m_{rr} > 0$ go to step 5.

If $m_{rr} \leqq 0$ and w_r is primal go to step 6.

If $^{m}{}_{rr} \leqq 0$ and w_r is dual define

$$J = J - \{r\}$$

and repeat Step 4.

STEP 5. Perform the pivot $\langle w_r, z_r \rangle$ and return to Step 6.

STEP 6. Perform the double pivot $\langle w_r, z_0 \rangle$, $\langle w_0, z_r \rangle$ drop the last row and return to Step 1.

This terminates the listing of the steps of the algorithm.

Before showing the finiteness of the algorithm a few comments are in order.

Let us choose, according to some rule to be specified later, some of the constraints and nonnegativity restrictions in our problem and form an index set from the indices identifying them. Denote it by I. Form a new problem by restricting the inequalities in the chosen constraints and nonnegativity restrictions to be satisfied as strict equalities. The set of feasible solutions of this new problem is a subset of feasible solutions of the original problem so it is natural to call it a restricted problem and denote it by $P(I)$. Using this notation the original problem is $P(\phi)$ where ϕ is a symbol for the empty set.

The algorithm just described works as follows: we start from a basic solution of the Kuhn-Tucker equations of P. By appropriate choice of I, we find a restricted problem $P(I_1)$ for which the above solution is a local minimum point. This can be seen from Theorem 7.

Having any restricted problem $P(I_k)$, where $I_k \neq \phi$, and a local minimum to it, we define a finite procedure the result of which is either (i) a new restricted problem $P(I_{k+1})$ and a local minimum to it or (ii) the fact that the objective function has no lower bound in the set of feasible solutions.

After initialization in Step 1 we find out whether the current basic solution is optimal or not. If it is not then according to Theorem 7 we can regard it as being optimal to a restricted problem $P(J)$ in which J contains all the nonbasic primal variables the complements of which are negative.

In Step 2 we add to the Kuhn-Tucker conditions of the restricted problem the appropriate constraint of the form of Equation (21). The remaining steps accomplish the task of finding a new restricted problem with an optimal solution that is better than the current one, or determining that the objective function is unbounded on the feasible region. In Step 3 we achieve feasibility in the restricted system and then in Step 4, we try to relax the auxiliary constraint by increasing λ from its original value 0. From the Corollary of Theorem 11 we know that an increase of λ implies a decrease in the value of the objective function, i.e., improves the current solution. If for some value of λ we lose feasibility then we try to change the current basis using one of the admissible pivot operations. In certain circumstances we cannot do so. According to Theorem 8 this difficulty can only arise in the case where we are going to lose dual feasibility. In such a case we permit the loss of dual feasibility and form a new restricted problem.

In summary, Step 3 produces a primal feasible complementary basic solution. For Steps 4, 5, 6, with $z_0 > 0$, according to Theorem 11, increasing λ will cause a decrease of the objective value: in case (i) indefinitely, in case (ii) the feasibility requirement bounds this increase. At the bound the effectiveness of the current basis ends, we have to change it either according to Theorem 8 in Step 5 or according to Theorem 9 in Step 6, whichever is appropriate. Finally, if none of them applies, we waive our feasibility requirement knowing that dual feasibility is not required at this stage.

We now prove convergence of the algorithm to an optimal solution in a finite number of iterations.

THEOREM 12: The algorithm described above will arrive at a local minimum point in a finite number of steps.

PROOF: According to Theorem 6 any complementary basic feasible solution in which the critical matrix is positive semidefinite or empty satisfies the second order optimality criteria, and as it is complementary and both primal and dual feasible it is also a Kuhn-Tucker point. In short it is a local minimum point.

The starting solution is a complementary basic solution with an empty critical matrix. The algorithm consists of principal pivot steps on diagonal elements or on two by two diagonal blocks only, thus every solution is complementary. The principal pivot operations in the algorithm are chosen in such a manner that they always maintain primal feasibility and they are made either on pivot elements specified in Theorem 8 or on two by two pivot blocks specified by Theorem 9. As a result, the critical matrix always stays positive semidefinite or empty.

Whenever a complementary primal feasible solution found is not dual feasible the algorithm does not stop. Thus the only thing that must be proven is the fact that the algorithm will stop after a finite number of steps.

According to the corollary of Theorem 11 any increase of λ implies the decrease of the value of the objective function. It is easy to see that as a result of Step 3 and every execution of Step 5 the maximal value of λ for which the current basic solution is primal feasible strictly increases. The number of bases to a given restricted problem is finite and max λ is strictly increasing thus no basis can be repeated. Therefore after a finite number of steps we reach Step 1 again and are at a lower value of the objective function.

From this and the fact that both the number of possible restricted problems and the number of bases in a given restricted problem is finite we can see that after a finite number of iterations we must reach the case when $J = \phi$ or determine that no solution exists. According to Theorem 7 this will be a local optimum to the original problem.

EXAMPLE:

$$\text{Minimize } q(X) = (-1, -2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1/2\,(x_1, x_2)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{Subject to } \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geqslant \begin{bmatrix} -4 \\ -5 \end{bmatrix}$$

$$x_1, x_2 \geqslant 0$$

Kuhn-Tucker conditions:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(x_1, x_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0; \qquad (v_1, v_2) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$x, y, u, v \geqq 0$$

The initial Kuhn-Tucker conditions for the example are put into tableau (0). Since the point $(x_1, x_2) = (0, 0)$ is primal feasible, we have a Kuhn-Tucker solution to the original problem with the additional restrictions that $x_1 = 0$ and $x_2 = 0$ (corresponding to the negative dual variables u_1 and u_2). Since (0, 0) is the only feasible point, we have a local minimum (the "critical" matrix is empty).

The restrictions are put into the form $x_1 + x_2 = \lambda$, $\lambda \geqq 0$ in tableau (0). At this point we are ready to attempt to move from the current restricted problem to a new restricted problem. According to the rules in step 3, we choose u_2 as the variable to leave the basis. Thus, after the initialization pivots $\langle u_2, z_0 \rangle$, $\langle w_0, x_2 \rangle$ we have tableau (2).

For $\lambda_c = 1$, u_1 becomes zero and will become negative for $\lambda > 1$. However, since the diagonal element corresponding to u_1 is nonpositive, we cannot pivot and therefore, following step 4, case (ii), we let u_1 go negative and continue increasing λ.

For $\lambda_c = 2$, the variable z_0 becomes critical. Thus, we pivot $\langle z_0, w_0 \rangle$ and arrive at tableau (3). Since $u_1 = -1$, and we otherwise have a feasible Kuhn-Tucker point, we will have a local minimum to the original problem with the additional restriction that $x_1 = 0$. The "critical" matrix is 1, and thus positive semi-definite.

In tableau (3) we have added the constraint $x_1 \leqq \lambda$, and are now ready to proceed to a new restricted problem. The initializing pivots $\langle u_1, z_0 \rangle$, $\langle w_0, x_1 \rangle$ result in tableau (5). v_1 becomes zero for $\lambda_c = 2$, and so we perform $\langle v_1, y_1 \rangle$ since the appropriate diagonal element is positive.

The next variable to become critical is v_2, for $\lambda_c = 3$. Since the desired pivot element is nonpositive, we perform the double pivot $\langle v_2, w_0 \rangle$, $\langle z_0, y_2 \rangle$ following case (ii) of step 4 and step 6. The resulting tableau (8) is a feasible Kuhn-Tucker point and also a local minimum point since the critical matrix is given by

$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

and is positive semidefinite.

For this example, the point $(x_1, x_2) = (3, 1)$ is also the global minimum.

Finally, we observe that in tableaux (2) and (5) an increase in λ results in a decrease in the value of the objective function since in both cases $z_0 > 0$ for $0 \leqq \lambda \leqq \lambda_c$, $\lambda_c > 0$.

	b.v.	1	λ	x_1	x_2	y_1	y_2	z_0	
	u_1	−1	0	−1	0	1	2	1	
	u_2	−2	0	0	1	1	−1	1	
(0)	v_1	4	0	−1	−1	0	0	0	
	v_2	5	0	−2	1	0	0	0	
	w_0	0	1	−1	−1	0	0	0	
				x_1	x_2	y_1	y_2	u_2	
	u_1	1	0	−1	−1	0	3	1	
	z_0	2	0	0	−1	−1	1	1	
(1)	v_1	4	0	−1	−1	0	0	0	
	v_2	5	0	−2	1	0	0	0	
	w_0	0	1	−1	−1	0	0	0	
				x_1	w_0	y_1	y_2	u_2	
	u_1	1	−1	0	1	0	3	1	
	z_0	2	−1	1	1	−1	1	1	$\lambda_c = 2$
(2)	v_1	4	−1	0	1	0	0	0	
	v_2	5	1	−3	−1	0	0	0	
	x_2	0	1	−1	−1	0	0	0	
	b.v.	1	λ	x_1	z_0	y_1	y_2	u_2	
	u_1	−1	0	−1	1	1	2	0	
	w_0	0	1	−1	0	0	0	0	$f(x) = -2$
(3)	v_1	2	0	−1	0	1	−1	−1	For $x_2 = 2$
	v_2	7	0	−2	0	−1	1	1	$x_1 = 0$
	x_2	2	0	0	0	−1	1	1	
				x_1	u_1	y_1	y_2	u_2	
	z_0	1	0	1	1	−1	−2	0	
	w_0	0	1	−1	0	0	0	0	
(4)	v_1	2	0	−2	0	1	−1	−1	
	v_2	7	0	−2	0	−1	1	1	
	x_2	2	0	0	0	−1	1	1	

				w_0	u_1	y_1	y_2	u_2	
	z_0	1	1	−1	1	−1	2	0	
	x_1	0	1	−1	0	0	0	0	
(5)	v_1	2	−1	1	0	1	−1	−1	$\lambda_c=2$
	v_2	7	−2	2	0	−1	1	1	
	x_2	2	0	0	0	−1	1	1	
		1	λ	w_0	u_1	v_1	y_2	u_2	
	z_0	3	0	0	1	−1	−3	−1	
	x_1	0	1	−1	0	0	0	0	
(6)	y_1	−2	1	−1	0	1	1	1	
	v_2	9	−3	−3	0	−1	0	0	$\lambda_c=3$
	x_2	4	−1	1	0	−1	0	0	
				v_2	u_1	v_1	y_2	u_2	
	z_0	3	0	0	1	−1	−3	−1	
(7)	x_1	3	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	
	y_1	1	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	1	1	
	w_0	−3	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	
	x_2	1	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	0	0	
				v_2	u_1	v_1	z_0	u_2	
	y_2	1	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	Local minimum
	x_1	3	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	
(8)	y_1	2	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$f(x)=-9$
	w_0	−3	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	
	x_2	1	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	0	0	

4. CONCLUDING REMARKS

In this section, we shall briefly discuss two points. First, some preliminary computational results will be used to compare the computational efficiency of the algorithm with existing algorithms for the convex quadratic programming problem. Second, we shall briefly mention computational aspects when we are finding a (local) minimum for the case where the objective function is concave.

In order to arrive at some initial conclusion as to the computational efficiency of the algorithm, an example problem was run with 15 variables, 15 constraints, and rank of the matrix C equal to 15. This size problem was run four times with the following variations:

(1) Q positive definite with three negative dual variables in the initial basic solution;

(2) Q positive definite with nine negative dual variables in the initial basic solution;
(3) Q positive definite with 15 negative dual variables in the initial basic solution;
(4) Q indefinite with three negative dual variables in the initial basic solution.

The results obtained are as given in Table 4–1.

TABLE 4–1. *Preliminary computational results for the local minimum algorithm*

Problem	Iterations	Time (sec)
Convex, neg. $u=3$	12	0.640
Convex, neg. $u=9$	13	0.630
Convex, neg. $u=15$	18	0.724
Nonconvex, neg. $u=3$	50	2.699

In order to compare these initial results with those obtained by other existing quadratic programming algorithms, we consider a study made by Moore and Whinston [4] in 1966. They studied the effects of size, rank, and number of negative dual variables in the initial basic solution on several different classes of algorithms. Basically, the algorithms tested in their study could be divided, according to relative efficiency, into three groups (see [4]). In Table 4–2, we have extracted from their study the results obtained for a problem with 15 variables, 15 constraints, and rank of the quadratic matrix equal to 15.

TABLE 4–2. *Results from appendix table 1, Moore and Whinston* [6]

(Problem size = 15 variables; rank of $Q=15$)

Group	Initial neg. $u=6$	Initial neg. $u=12$
1	5.17 iterations (average)	5.4 iterations (average)
2	6.73	10.25
3	11.05	29.15

Although any conclusions at this point must be tentative at best, the local minimum algorithm does not compare too unfavorably with those in group 2, and seems more efficient than those algorithms in group 3—particularly as the number of initially negative dual variables increases. No attempt is made to compare the relative speed of the algorithm with the others since the problems in Moore and Whinston's study were solved by a different (and older) hardware system than that used to program the local minimum algorithm.

Further efforts to judge the efficiency of the local minimum algorithm have not been made at this point since the primary interest in the algorithm is not its ability to solve convex problems efficiently, but the fact that it can solve nonconvex problems. A number of additional runs have been made on smaller problems designed to test particular aspects of the algorithm with good results, however.

One computational aspect of the local minimum algorithm that may be of interest is that in the concave case (where the matrix Q is negative definite), no diagonal pivots are made in reaching the optimal solution. All iterations consist of a pair of off-diagonal pivots. Advantage of this fact can be taken for known concave problems to reduce data storage that is otherwise required in the general

nonconvex problem where diagonal pivots may be necessary.

In conclusion, we remark that, although we have not dealt with the problem of degeneracy in this paper, we are confident that all problems relating to degeneracy can and will soon be solved.

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SIMULTANEOUS EQUATION PRODUCTION FUNCTIONS (COBB-DOUGLAS TYPE) FOR DECISIONS PERTAINING TO SEA-BASED TACTICAL AIR RESOURCES*

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ABSTRACT

This paper proposes the use of a set of simultaneous, modified Cobb-Douglas production functions to determine the "best" mix of aircraft inputs (men, spare parts, support equipment, etc.) to achieve certain levels of output. The simultaneous equations are "solved" by the use of proxy values in an iterative search process. Tests of this production function model indicate that real world data fits quite well (R^2 ranges from 0.884 to 0.951) and the residuals do not appear to have any significant bias.

The aircraft production functions for several types of aircraft (F–4, A–7, A–6, A–4) have been maximized for a fixed budget and specific constraints. A potential increase in output of up to 10 percent appears possible. In general the marginal value of spare parts surpasses the other inputs.

I. INTRODUCTION

With today's increasing cost of defense, the need for improved analytical aids to assist top management in making allocation decisions concerning the "best" mix of military aircraft resources (aircraft, men, spare parts, support equipment, etc.) is quite clear. The investment cost of a Navy attack aircraft has risen from $60,000 (World War II) to approximately $200,000 (Korea) to in excess of $3,000,000 today. During the same time span maintenance cost (60 percent) has not risen as fast. It is likely that the "best" mix of resources to produce today's required output is different from what, intuitively, would have been an effective allocation policy a short time ago.

II. THE PROBLEM AND THE JUSTIFICATION TO USE A COBB-DOUGLAS FUNCTION

The problem is to develop a methodology which will serve as an aid to decision makers so that they can arrive at a "better" allocation of tactical aircraft carrier inputs to achieve specific outputs (effective sorties and ready hours).**

Historically, the output of military aircraft has been difficult to define and quantify. Ferguson [3], in 1963, reported that due to a lack of recorded data, an economist cannot properly evaluate the output of military aviation. Several years later (1971), Enthoven and Smith [2] reported that the military can

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*This research was undertaken as part of a George Washington University doctoral dissertation that was completed in March 1972.

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**There is no agreement on any specific outputs or measures of effectiveness for tactical aircraft. They range from measurable items such as effective sorties, ready hours, and loiter time, to such nonquantitative considerations as raising the national level of technology and providing an industrial base for mobilization. For each specific type of aircraft, the dynamic MOE of sorties and the static MOE of ready hours will be used. Before different aircraft can be compared, however, the MOE units must be normalized for payload (bombs), firepower (air-to-air weapons), and the expected probability of mission success.

quantify the aviation output parameters through the units of potential sorties (adjusted for lethality of ordnance), loiter time, and air crew proficiency. Donaldson and Blake [1] consider the military output of fighters to be effective missions (measured in sorties) and ground alert time. Hitch and McKean [5] report that military aircraft generate two primary outputs, flying hours or sorties and aircraft in commission (usually measured in ready hours), in a production function situation "analogous to others encountered by economists in industry." Sutton [7] reports that based upon the then recent (1966) Navy maintenance data, a Cobb-Douglas* or CES (Constant Elasticity of Substitution) type of production function relationship exists between worldwide reports (ship and shore) of U.S. Naval Aircraft inputs and the output of readiness (measured in aircraft ready hours). As a result of testing for neatness of fit and the examination of residuals (error between the model and the actual observation) both of these production functions are acceptable as a representation of the real world. However due to computation ease and computer savings in time the Cobb-Douglas function was favored over the CES.

Sutton was able to obtain recorded observations for only one output (ready hours or hours that an aircraft is "up" and ready for flight) and three inputs (aircraft, men, and spare parts).

Gilster and Woodman [4], when investigating the relationship between the Air Force's use of labor and capital to generate output (flying hours), determined that an exponential (Cobb-Douglas) relationship appeared to exist between the inputs and the outputs.

To date all known research points toward a Cobb-Douglas production function type situation. Due to the limited recorded data of the military aircraft maintenance/flying process, none except Sutton have been able to quantify the elasticities of the actual inputs. In addition, no specific research has been done to establish what relationships may exist between the various management policies and output, or how this production function relationship may change under the constraints found aboard ship.

Due to improved military maintenance recording and reporting procedures, I am now able to record simultaneously in actual units, month by month, for each Navy squadron the outputs of sorties (U_1) and aircraft ready hours (U_2). At the same time, the four inputs of aircraft (W_1), maintenance personnel (W_2), actual units of support equipment in use (W_3), and total spare parts available to meet demands (W_4) were recorded for specific maintenance and operating policies of (X_k) *and* (Y_l).

III. MODEL AND OBJECTIVE FUNCTION

Simultaneous equation, Cobb-Douglas type, of a joint production function of the following type is proposed:

$$U_1 = {}_1\alpha_0 \, {}_1W_1^{{}_1\alpha_1} \, {}_1W_2^{{}_1\alpha_2} \, {}_1W_3^{{}_1\alpha_3} \, {}_1W_4^{{}_1\alpha_4} \epsilon_1$$

$$U_2 = {}_2\alpha_0 \, {}_2W_1^{{}_2\alpha_1} \, {}_2W_2^{{}_2\alpha_2} \, {}_2W_3^{{}_2\alpha_3} \, {}_2W_4^{{}_2\alpha_4} \epsilon_2,$$

where

U_i = the multiproduct outputs (U_1 = sorties, U_2 = ready hours),
${}_i\alpha_0$ = a scaling efficiency or technical change factor,
${}_iW_j$ = inputs (aircraft, men, support equipment, spare parts),
${}_i\alpha_j$ = elasticity with respect to the ${}_iW_j$ input,
ϵ_i = the multiplicative error term.

*A linear homogeneous function that has the following form: $P = bL^{k}K^{j}$, where P means product, L means labor input, K capital input, b a constant, and k and j are the elasticities with respect to labor and capital.

But there is a negative correlation between the joint products of ready hours and the sortie rate. Increasing the sortie rate tends to decrease the ready hours. As the number of sorties goes up, the time the aircraft is down increases for support actions and unscheduled maintenance actions. High availability or a greater number of ready hours means a higher number of flights could be flown (you have to have an aircraft up before it can be flown); ready hours available will go down or be consumed when the increased sorties are flown. Analysis of the data for the five types of aircraft observed (F–4, A–7, A–6, A–4, and E–2) in this research indicates that the "best" fit is approximately an arc of a circle (see Figure 1).

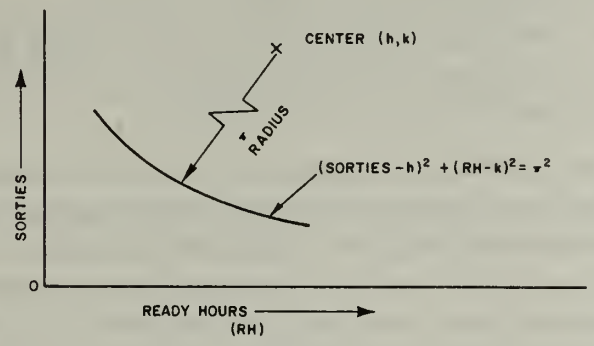

FIGURE 1.

In theory the two outputs could be treated as a single production function. This has been tried and such a single Cobb-Douglas function did not have the same high neatness of fit that the simultaneous equation had at the extreme range of observations. For these simultaneous outputs, the two products are assumed to have a region where they act as substitutes for each other; therefore the lack of independence between these two equations should not be serious.

This means if all inputs (W_1, W_2, W_3, W_4) are held constant and sorties are increased, ready hours (U_2) above the required residual level (established by policy to meet potention future demands) act as a substitute; or some Δ increment of (U_2) is an input of U_1.

The objective of the optimization model is to maximize sorties for a fixed ready hour policy. Holding ready hours constant ($\bar{U}_2$), for any set of maintenance and flying policies, the net production function of the two simultaneous equations, to be maximized, is of the type:

$$(1) \qquad \bar{U}_1 = \left[{}_1\alpha_0 \, {}_1W_1^{{}_1a_1} \cdots {}_1W_4^{{}_1a_4} + \frac{dU_1}{dU_2} \Delta \left({}_2\alpha_0 \, {}_2W_1^{{}_2a_1} \cdots {}_2W_4^{{}_2a_4} \right) \right]; Y_k, Y_l,$$

where

$\bar{U}_1$ = potential sorties,

$\frac{dU_1}{dU_2}$ = the derivative or slope of U_1 to U_2,

Δ = the potential absolute increase in ready hours above or below the required policy level or base case ($\bar{U}_2$),

Y_k, Y_l = Two types of policy, a combined tactical decision or scenario dimension (Y_k) (i.e., flying shall be continuous in 1¾-hour cycles) with a maintenance decision (Y_l) (fix all discrepancies as occurring, fly until a failure occurs, etc.)

$W_j = {}_1W_j + {}_2W_j$.

Constraints are:

$$W_j \geq 0 \text{ (feasible region)}$$
$$d_j \leq W_j \leq b_j \text{ (resource constraints)}$$
$$\sum_{j=1}^{n} C_j W_j \leq C \text{ (cost constraint),}$$

where

C_j — is the unit cost of each resource in dollars, by type (j).

C — is the upper dollar budget limit of all resources for a unit of time.

d_j & b_j — represent lower limit and upper limit, respectively, of the physical constraints on the total weight or size (ft^2) of resources, by type (j).

To consider further the Cobb-Douglas simultaneous production function, it should be noted that I have not fully resolved the issue of the degree of substitutability that exists between the outputs and the recording problems of measuring the inputs. Going back to the observed world, I am able to record independently the various levels of U_1 and U_2 for specific total levels of inputs (W_j). In addition, I do have a relationship between each input vector (${}_1W_1, {}_2W_1; {}_1W_2, {}_2W_2; \ldots; {}_1W_4, {}_2W_4$) and the outputs. For example, I observe the total labor that goes jointly to produce both U_1 and U_2, but not the amount of labor just to produce U_2. In summary, I observe the "rim" or external input and output values of the following two simultaneous equations:

$$U_1^{\#} = {}_1\alpha_0 \, {}_1W_1^{{}_1\alpha_1} \, {}_1W_2^{{}_1\alpha_2} \, {}_1W_3^{{}_1\alpha_3} \, {}_1W_4^{{}_1\alpha_4} \, \epsilon_1$$

$$U_2^{\#} = {}_2\alpha_0 \frac{{}_2W_1^{{}_2\alpha_1}}{W_1^{\#}} \frac{{}_2W_2^{{}_2\alpha_2}}{W_2^{\#}} \frac{{}_2W_3^{{}_2\alpha_3}}{W_3^{\#}} \frac{{}_2W_4^{{}_2\alpha_4}}{W_4^{\#}} \epsilon_2,$$

where

$\#$ = indicates a "rim" value that is actually observed

${}_1W_1$ = the cell input of W_1 to assist in output 1

.

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.

${}_2W_4$ = the cell input of W_4 to assist in output 2

${}_1\alpha_1$ = elasticity with respect to the ${}_1W_1$ input

.

.

.

${}_2\alpha_4$ = the elasticity with respect to the ${}_2W_4$ input

${}_1\alpha_0$ = the technology change factor with respect to output 1

${}_2\alpha_0$ = the technology change factor with respect to output 2.

The estimation problem is to solve the above equations simultaneously for the unknown "cell" values (${}_1\alpha, {}_1\alpha_1, \ldots, {}_2\alpha_4$, and ${}_1W_1, \ldots, {}_2W_4$).

A procedure to quantify the values for the unknown cells or to obtain estimates of the α_j's (elasticities of the inputs) is as follows:

(a) First, rewrite the production function in natural logarithms to establish linear equations.

(b) Second, estimate the initial values for ${}_1W_1$, ${}_1W_2$, etc., such that they satisfy—

$$W_1 = {}_1W_1 + {}_2W_1, \text{ with all } W_j > 0.$$

(c) Third, with the above initial values obtain approximate estimates for the α_j's by a likelihood estimate methodology [6]. This is the end of the first iteration.

(d) Fourth, update the W_j's by a quadratic algorithm (6).

(e) Fifth, update the α_j's by the likelihood estimate methodology. This is the end of the second iteration.

(f) Continue this iterative process until a local minimum has been reached.

(g) Last, test for convergence, or near convergence, to a "best" local solution by varying the initial values used in step (b) over a wide range in order to increase the probabilities that an approximate global solution has been reached.

An illustration of this methodology can best be shown by using a smaller example to demonstrate the general case relationships.

In the small case I have two inputs, capital (K) and labor (L), each with its coefficient of elasticity (α and β, respectively).

In this case we have

$$\ln U_1 = \ln \alpha_0 + \alpha_1 \ln K_1 + \beta \ln L_1 + \ln \epsilon_1$$

and

$$\ln U_2 = \ln \alpha_0' + \alpha_1' \ln K_2 + \beta' \ln L_2 + \ln \epsilon_2,$$

and I observe $U_1, U_2; K = K_1 + K_2; L = L_1 + L_2$ for each time period.

In addition, I must make a few mild assumptions. However, there is strong empirical evidence that these assumptions and the resulting model accurately reflect real world experience. I assume:

(a) The distribution of U_i $(i = 1, 2)$ is approximately normal with an expected value of U_i that is a linear function of K_i and L_i and has a variance that is independent of K_i and L_i.

(b) The values taken on by K_i and L_i are predetermined.

(c) The successive values of U_i are independent from the prior values.

(d) $\ln \epsilon_{ij}\,(i = 1, 2), j = 1, 2, \ldots n$, are normally distributed with $E(\epsilon_{ij}) = 0$ and Cov $(\epsilon_{ij}, \epsilon_{ik}) = 0$ for all $j \neq K = 1, 2 \ldots n$, and $V(\epsilon_{ij}) = \sigma^2$ for all j.

The Derivation of a Related Likelihood Function

To determine the maximum likelihood (L) estimates of α_0, α_1, and β, we have:

$$\text{Max } L = \frac{1}{(2\Pi)^{T/2}\sigma_1^T} e^{-\left(\frac{1}{2\sigma_1^2}\right)\left(\sum_{t=1}^{T} [\ln U_1^t - \ln \alpha_0 - \alpha_1 \ln K_1^t - \beta \ln L_1^t]^2\right)},$$

where T = the number of observations obtained over time.

The partial derivative of $\ln L$ with respect to σ_i is of the type:

$$\frac{\partial \ln L}{\partial \sigma_1}=\frac{T}{\sigma_1}-\frac{1}{2}S\times\frac{(-2)}{\sigma^3}=0,$$

which becomes

$$\sigma_1^2=\frac{S}{T},$$

where

$$S=\sum_{t=1}^{T}[\ln U_1^t-\ln \alpha_0-\alpha_1 \ln K_1^t-\beta \ln L_1^t]^2.$$

So the minimized natural logs of the likelihood function for a_0, α_1, β is proportional to

$$\ln \sum_{t=1}^{T}[\ln U_1^t-\ln \alpha_0-\alpha_1 \ln K_1^t-\beta \ln L_1^t]^2. \tag{2}$$

Now the determination of the maximum values of α_0, α_1, β, etc. for the given K_i's and L_i's is straightforward. Of course, the maximum likelihood estimators may have a bias when small samples are used. Analysis of the residuals from the actual data has established limits on the bias problem.

The Derivation of a Related Quadratic Solution

Unfortunately, we observe only the total values of K and L, *or* $K=\Sigma K_i$ and $L=\Sigma L_i$'s. But the individual values for the K_i's and L_i's can be determined by a sequential technique, moving toward a minimum value of Equation (2). When the "best" approximate values have been established for α_0, α_1, β etc., then the objective function (1) can be maximized, subject to the constraints.

To solve for the K_i's start with initial estimates for K_1^t, K_2^t, L_1^t, L_2^t, that will satisfy the total observed values.

Then with these initial, or proxy, values, estimates for α_0, α_1, β, α_0', β' can be obtained by regression analysis.

With the estimates of α_i's and β_i's, the improved values of K_1, K_2, L_1, L_2 for the next iteration now exist where the sum of the two error terms (ϵ_1, ϵ_2)—

$$\ln \sum_{t=1}^{T}(\ln U_1^t-\ln \alpha_0-\alpha_1 \ln K_1^t-\beta \ln L_1^t)^2+\ln \sum_{t=1}^{T}(\ln U_2^t-\ln \alpha_0'-\alpha_1' \ln K_2^t-\beta' \ln L_2^t)^2$$

is a minimum.

However, if we let $k_1^t=\frac{K_1^t}{K^t}$ (in order to have $0<k_1<1$) and $\tilde{k}_1^t=1-k_1^t$, and we let

$$\tilde{U}_1^t=\ln U_1^t-\ln \alpha_0-\alpha_1 \ln K^t-\beta \ln L_1^t$$

$$\tilde{U}_2^t=\ln U_2^t-\ln \alpha_0'-\alpha_1' \ln K^t-\beta' \ln L_2^t,$$

hen we desire the minimum for

$$\ln \sum_{t=1}^{T} (\tilde{U}_1^t - \alpha_1 \ln k_1^t)^2 + \ln \sum_{t=1}^{T} (\tilde{U}_2^t - \alpha_1' \ln \bar{k}_1^t)^2. \tag{3}$$

Taking the partial derivatives of (3) with respect to k_i and setting this equal to zero, we have

$$\frac{\partial}{\partial k_i} = \frac{2(\tilde{U}_1^t - \alpha_1 \ln k_1^t)}{\sum_{t=1}^{T} (\tilde{U}_1^t - \alpha_1 \ln k_1^t)^2} \left(\frac{-\alpha_1}{k_1^t}\right) + \frac{2(\tilde{U}_2^t - \alpha_1' \ln \bar{k}_1^t)}{\sum_{t=1}^{T} (\tilde{U}_2^t - \alpha_1' \ln \bar{k}_1^t)^2} \left(\frac{\alpha_1'}{\bar{k}_1^t}\right) = 0. \tag{4}$$

But a series expansion of $\ln k_1$ would be

$$(k_1 - 1) - 1/2(k_1 - 1)^2 + 1/3(k_1 - 1)^3 - 1/4(k_1 - 1)^4 + 1/5(k_1 - 1)^5 \ . \ . \ .$$

And if

$$a = \sum_{t=1}^{T} [\tilde{U}_1^t - \alpha_1 (k_1^t - 1)]^2 \tag{5}$$

and

$$b = \sum_{t=1}^{T} [\tilde{U}_2^t - \alpha_1' (\bar{k}_1^t - 1)]^2, \tag{6}$$

using the series expansion and multiplying the terms, we have:

$$\frac{-\alpha_1 \tilde{U}_1^t + \alpha_1^2 (k_1^t - 1)}{a k_1^t} + \frac{\alpha_1' \tilde{U}_2^t + \alpha_1'^2 k_1^t}{b(1 - k_1^t)} = 0.$$

Then by transposing and collecting terms, this becomes a quadratic of the form:

$$k_1^{t2} (a\alpha_1'^2 - b\alpha_1^2) + k_1^t [2b\alpha_1^2 + b\alpha_1 \tilde{U}_1^t + a\alpha_1' \tilde{U}_1^t] + (-b\alpha_1^2 - b\alpha_1 \tilde{U}_1^t). \tag{7}$$

The values for a and b change little between iteratives, as we converge to a solution, if T is sufficiently large (> 20). Therefore, the values of a and b from the prior iteration can be used for the present iterative; and we have, for each step, a near constant value for a and b for any specific iteration.

This quadratic (7) now has only one unknown, k_1, for which an estimated solution can be obtained. In a similar fashion, L_1 (labor) can be estimated.

With these revised values for K_1, K_2, L_1, L_2, updated regression estimates can now be made for x_0, α_1, β. This, in turn, updates the values of K_1, K_2, L_1, L_2. Thus, the sequential process continues until we have the "best" values for α and β, such that the sum of the squares is a minimum. To test the possibility that this "final" solution may be but a local value near absolute minimum, different initial estimates for K_1, K_2, L_1, L_2 were made. If several local minima are established, a fairly ex-

haustive search was made to determine which of several subsolutions are unwanted stationary points and which are the near "true" minimum values.

In the general case, if we let

$$ {}_1Z_1^t = \ln U_1^t - \ln {}_1\alpha_0 - {}_1\alpha_2 \ln {}_1W_2^t - {}_1\alpha_3 \ln {}_1W_3^t - {}_1\alpha_4 \ln {}_1W_4^t $$

and

$$ {}_1Z_2^t = \ln U_2^t - \ln {}_2\alpha_0 - {}_2\alpha_2 \ln {}_2W_2^t - {}_2\alpha_3 \ln {}_2W_3^t - {}_2\alpha_4 \ln {}_2W_4^t. $$

Then via the method that was used to develop Equation (3) the problem becomes one of

$$ \text{Min} = \ln \sum_{t=1}^{T} ({}_1Z_1^t - \alpha_1 \ln {}_1W_1^t)^2 + \ln \sum_{t=1}^{T} ({}_1Z_2^t - {}_2\alpha_1 \ln {}_2\bar{W}_1^t)^2. $$

Then by taking the partial derivatives with respect to w_1', and collecting the terms, the quadratic becomes one of the type:

$$ w_1'^2({}_2\alpha_1^2 - {}_1\alpha_1^2) + w_1'(2{}_1\alpha_1^2 + {}_1\alpha_1 {}_1Z_1 + {}_2\alpha_1 {}_1Z_2) + (-{}_1\alpha_1^2 - {}_1\alpha_1 {}_1Z_1) = 0. $$

Again, by a series of sequential steps, a best minimum solution can be approached which in turn will establish the "best" values for the α_i's. Returning to the objective function (1), the output is maximized, subject to the constraints.

IV. SOLUTION TO THE PROBLEM

A program was written for use on a Control Data (CDC) 3800 system in FORTRAN IV language. Thirty random observations* were used, and the number of iterations was usually 30; but, for diagnostic runs, it was extended to 100. Sensitivity of starting points was tested over a wide range.

As the quadratic methodology type search moves through the feasible region toward the boundary, an increasing number of observations have such recorded values that the quadratic solution for one or more of the inputs is of the type ${}_1W_j^t = 0$ or ${}_2W_j^t = 0$. This is not a "true" indicator of the real world allocation of resources. This would mean that all the inputs are just producing one of the outputs, neglecting the other. Under this unrealistic allocation of inputs, the quadratic methodology has its greatest bias and the iterative search for a minimum value tends to become unstable for at least one iteration. On the next iteration, and on a zigzag vector *away* from the global solution, the process starts again to converge to a new local. In the case of the F–4 aircraft data, the first local minimum was reached via a zigzag approach at about the 12th iteration. After that, near similar locals were obtained in the 16th, 23rd, 29th, and 35th iterations. The F–4 data have been run up to 100 iterations without any noticeable change in this pattern. Cycling may be occurring after the 25th iteration.

Many different starting points have been used and they result in obtaining approximately the same local "best" solution.

Considering the wide range of different starting points used and the fact that the search was con-

*Taken from a total population of all at-sea squadron maintenance actions that occurred between July 1968 and October 1970. The total population consisted of 178 observations.

tinued for up to 100 iterations, it is reasonable to assume that the average lower value obtained is a near convergence to an absolute minimum.

In this F–4 aircraft case, the minimum value iterations gives the following results:

	α_1	α_2	α_3	α_4	
Sorties (U_1)	0.044	0.033	0.025	0.025	Σ 0.107
Ready hours (U_2)	1.090	—	0.420	0.120	Σ 1.630.

Three random samples in all were drawn from the total F–4 observations, each with a different average level of ready hours. For these samples, the coefficient of determination (R^2) ranged from 0.884 to 0.926.

A sequential F-test of the variables for the sortie output was made to determine if each variable appeared to contribute sufficiently. Since the F values in all cases exceed the critical values (at the 0.95 level of confidence), each of these variables appears worthwhile. The residuals were analyzed for three types of possible error. No sampling bias error could be detected. The variance proportion of the residual error was about 0.263, and the covariance proportion was about 0.737. A plot of residuals indicated some difficulty with outlier observations. Keypunch errors and errors in the interpretation of reporting requirements probably account for the majority of the recorded maintenance outliers.

The labor elasticity for ready hours ($_2\alpha^2$) tended to be negative. But negative elasticities are meaningless. For the observed range of inputs, it is improbable that output would actually decrease with an increase in input. Finally, the degree to which $_2\alpha_2$ tended to be negative was not statistically significant. Although this tendency for the wrong sign is a warning of possible incorrect specification, the model is empirically preferred. Of course, one does not choose just the equation with the highest R^2. The included inputs (W_1 to W_4) are based on prior research, such as done by Gilster and Sutton, observations and theoretical considerations. The model appears to be reasonably specified and explains most of the variation observed.

Using the objective function (1) and the basic policy case (X_1, Y_1) (which is cyclic flying operations for 12 hours per day, and the fixing of all discrepancies as occurring), we have an average ready hour residual of 400 hours/aircraft/month, or about 56 percent availability. The value for $\frac{dU_1}{dU_2}$ is 0.546. Thus, for the F–4, the objective function becomes:

$$\bar{U}_1 = 187.52\ (_1W_1)^{.044}(_1W_2)^{.033}(_1W_3)^{.025}(_1W_4)^{.025} + |.546|\,\Delta(3.21)\,(_2W_1)^{1.09}(_2W_2)^{0.0}(_2W_3)^{.420}(_2W_4)^{.120}$$

$$= 270.95 \text{ sorties/month.}$$

If all resources are increased by 10 percent, holding $RH = 400$ constant, we have:

$$\bar{U}_1 = 187.52(3.3)^{0.044}(104.5)^{0.033}(55)^{0.025}(19.8)^{0.025}$$

$$+ |0.546|\,(3.21)\,(9.9)^{1.09}(60.5)^{0.0}(40.7)^{0.420}(67.1)^{0.120} - (262.66)|0.546|$$

$$= 298.504, \text{ or an increase of } 27.552 \text{ sorties or } 10.168 \text{ percent.}$$

Thus, for a 10-percent increase in inputs, about a 10-percent change (10.168 percent) in outputs or the summation of the elasticities is $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\frac{dU_1}{dU_2}(\alpha_1'+\alpha_2'+\alpha_3'+\alpha_4')\approx 1$. For this specified policy ($RH=400$), we have almost constant returns to scale as resources are increased.*

Using the calculated values for the elasticities, we can compare the below marginal physical products to their price, which gives a ranking of the possible outputs in relation to their basic cost. The ranking for a 10-percent shift in inputs is:

$$W_3 \text{ (support)}=\frac{6.886}{(8.7)\ (0.666)}=1.188 \text{ sorties/thousand dollars } (K)$$

$$W_4 \text{ (spares)}=\frac{2.454}{(7.9)\ (0.869)}=0.357 \text{ sorties}/K$$

$$W_1 \text{ (aircraft)}=\frac{17.319}{(1.2)\ (46.2)}=0.312 \text{ sorties}/K$$

$$W_2 \text{ (men)}=\frac{0.894}{(14.5)\ (0.466)}=0.132 \text{ sorties}/K.$$

Then, if the constraints of the F–4 objective function are

$$9\leq W_1\leq 14; \qquad 140\leq W_2\leq 165; \qquad 70\leq W_3\leq 125; \qquad 70\leq W_4\leq 150,$$

and

$$C_1W_1+C_2W_2+C_3W_3+C_4W_4\leq 748.6 \text{ } K/\text{month},$$

a "best" mix equal cost squadron, as determined by heuristic programming, is:

$$\bar{U}_1=187.53(2.83)^{0.044}(88.7)^{0.033}(71.8)^{0.025}(20.0)^{0.025}$$

$$+|0.546|\ [3.211(8.50)^{1.09}(51.3)^{0.0}(53.2)^{0.420}(70.5)^{0.120}-262.66]$$

$$=289.34 \text{ sorties}.$$

This is an increase of 18.39 sorties or 6.8 percent over the base case (289.34−270.95).** This increase occurs with the following shifts in basic inputs:

-0.67 aircraft; -10.0 men

$+38.0$ units of support; $+11.5$ units of spare parts.

For the constraints of the situation, we have suboptimized the F–4 mix of inputs.

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*All of the types of aircraft examined had near constant returns to scale except for the A–6 attack aircraft. In this case, the sum of the elasticities was about 0.820 for a 12-hour flying day; but it was postulated that this would increase to ≈ 1.0 for a 16-hour flying day.

**The increased output for a "best" mix varied from a high of + 9.6 percent (A–7 aircraft) to a low of 2.6 percent (A–4 aircraft).

Variations in Policy

Recall that the summation of the elasticities was about 1.0 for a specified policy level of flying. However, if the flying hour day is increased from 12 to 16 hours per day and the ready hours decreased to $RH = 300$, then the output will increase as high as 300.886 sorties, or an additional increase in output of + 0.879 percent due to a change in flying policy. This means for a 10 percent increase in inputs, *economies of scale* (11.047 percent output) will occur.

In a similar manner, if the flying day were decreased, holding all inputs constant (with an increase in readiness of $RH = 500$), the output would decrease and *diseconomies of scale* would occur.

If the maintenance policy is changed to progressive (on condition) scheduled maintenance and the deferring of flight discrepancies for batch processing, total labor savings of 16.9 man months occur (about 10 percent of the labor force). There will also be an increase in readiness (expert opinion estimates at least a 5-percent increase). This budget saving could purchase sufficient support equipment for an increase of 1.54 percent in output. The potential increased readiness might give an additional + 2.64 percent of output.

Thus, while variations in policy (maintenance and flying) may not have a dominate effect on output, they do increase somewhat the output and change the economies of scale of the production function.

Sensitivity to Cost Changes

Recall that a "best mix" equal cost F–4 squadron had a maximum output of 289.34 sorties at $RH = 400$. This corresponds to increasing the use of resources that have a high ratio of MPP/Cost and decreasing those with a low MPP/Cost. The cost data in this case is the basic Department of Defense recorded expenses associated with each of these inputs. In addition, several other "bundles" of cost were considered for each input. Aircraft costs were considered with and without attrition, investment cost was considered with and without the cost of space aboard an aircraft carrier, and the present value (PV) concept of cost was used.

The support (W_3) input was most robust and the marginal value of this input dominated, regardless of the cost variation. The W_4 (labor) input was always last! The other two inputs (W_1, W_2) were sensitive to price changes.

Average Versus Marginal Cost

Recall that for a fixed budget of 748.6 K/month, the "best" mix produced an output of 289.34 sorties. The average cost (AC) per sortie is 2.587. But the marginal cost (MC) to generate additional sorties by the use of each possible input is the reciprocal of the MPP_{W_j}/C_j, which is:

$MC_1 = 3.205$ using aircraft at ${}_2C_1$ cost

$MC_2 = 7.576$ using men at ${}_1C_2$ cost

$MC_3 = 0.842$ using support equipment at ${}_1C_3$ cost

$MC_4 = 2.801$ using spare parts at ${}_1C_4$ cost.

Since MC for $W_3 < AC$, it would be logical to increase this input until $MC = AC$, provided $W_3 \leq 125$ and if the budget can be exceeded. Unfortunately, this factor of production is restricted already by the

upper constraint. Therefore, the AC of 2.587 is the best average cost obtainable within the defined constraints. Note, however, that MC_4 is only a small amount above the best AC, and the constraints of the situation ($W_4 \leq 150$) permit an additional 59.5 units of spare parts. This means an additional 16.06 sorties or that 5.6 percent of output can be purchased with only a minimal increase in average cost if additional sorties are required. This may be an attractive alternative, say compared to new ship/airwing construction, for obtaining additional output.

V. CONCLUSIONS AND IMPLICATIONS

The Cobb-Douglas (C-D) type model appears to be reasonably specified and explains most of the observed variation. The maximum likelihood estimate used to obtain the C-D elasticities does not appear to have any sampling bias. The coefficient of determination (R^2) varied from 0.884 to 0.951 for the five types of aircraft examined (F–4, A–6, A–7, A–4, and E–2). Analysis of the residuals indicated the largest proportion of error was of the covariance type. This is probably mainly due to limitations in the accuracy of the maintenance data.

With the exception of the A–6, the sea-based aircraft production process appears to operate at near constant economies of scale. However, if the flying day is increased (all other inputs held constant), economies of scale should take place. In a constant budget situation, variations in both the flying hours (up to about 16 hours per day) and maintenance policy may increase the outputs by as much as 14 percent.

The aircraft production function for each type of aircraft can be maximized for a fixed budget and specified constraints. Under these conditions, the potential increased output varied from a + 2.65 percent (A–4 and the oldest aircraft) to a + 9.6 percent (A–7). The ratio of the MPP_j/Cost_j gives a good indication of which inputs should be substituted for each other. Usually, the marginal value of spare parts (W_4) surpassed the other inputs. In the case of the F–4, the marginal value of support equipment was dominant. In general, labor (W_2) (on the margin) had the lower priority.

This model permits the decision maker to gain insight into which factors of production should be substituted for each other through the general least-cost substitution principle. The maximizing of the C-D objective function for various levels of the budget clearly indicates that a better allocation of inputs—aircraft, men, support, and spares—is possible at the squadron level. Compared to the Navy's current slower review process, this methodology offers promise of rapidly arriving at a "best" allocation arrangement for newer types of aircraft.

The C-D model shows how and when changes in factor prices induce further substitution and how the average cost (AC) compares to the marginal cost (MC) for various capacity restrictions and levels of policy. Since the MC for some factors is definitely below AC, it seems logical to conclude that these relatively low marginal cost inputs are under-utilized. Finally, the model appears logical and all of the variables are supported by empirical evidence which reflects real world experience. Predictions based upon this model merit serious consideration.

The potential practical applications of this methodology are multifold. It is a method to increase potential output for a fixed budget, a way of exposing under-utilized resources, or a way to model the implications of price changes for the factors of production (i.e., all-volunteer armed forces pay rates). This approach is equally applicable to the airline industry and to military air operations—both desire to maximize the potential output within their capacity restrictions.

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A TWO PRODUCT DYNAMIC ECONOMIC LOT SIZE PRODUCTION MODEL WITH EITHER-OR PRODUCTION CONSTRAINTS

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ABSTRACT

This paper considers the production of two products with known demands over a finite set of periods. The production and inventory carrying costs for each product are assumed to be concave. We seek the minimum cost production schedule meeting all demands, without backlogging, assuming that at most one of the two products can be produced in any period. The optimization problem is first stated as a nonlinear programming problem, which allows the proof of a result permitting the search for the optimal policy to be restricted to those which produce a product only when its inventory level is zero. A dynamic programming formulation is given and the model is then formulated as a shortest route problem in a specially constructed network.

I. INTRODUCTION

This paper extends the production inventory model of the type originally formulated by Wagner and Whitin [5], and later extended by Eppen, Gould, and Pashigian [1], and Zangwill [6]. The original model, [5], considers a single product with known demands over N periods. It assumes constant, over time, linear production cost, and variable production set up and holding costs. For this model the authors exhibit a particularly efficient dynamic programming formulation. They obtain results of a "planning horizon" nature which insure, under certain conditions, that the optimal solution for the first K periods remains optimal for all problems involving a greater number of periods. The determination of the recursive relation value is simplified by using this result. Eppen, Gould, and Pashigian [1] extended this single product model by allowing a nonconstant (over time) linear production cost. They obtained similar "planning horizon" results with slightly more complex conditions for the existance of the planning horizon and computational simplifications. Zangwill [6], extended the model to allow for general concave production and holding costs. He also extended the model to allow for the case of backlogging, showing that concave shortage costs reduce the problem to the solution of an elegant set of two recursive relations. He also formulated the problem as a single-source, minimum-cost transshipment network with concave arc costs. All of these models considered a single product.

In this paper we consider two products with the constraint that at most one product can be produced in a single period. We assume concave production and holding costs and do not allow backlogging.

In section II we present the definition of the model and nonlinear programming formulation. This

model is used in section III to prove a result which is used in section IV to formulate the problem as a dynamic programming relationship. Section V gives a network representation of the problem which allows calculation of the optimal policies as a shortest route network problem. Some reductions in computing time are noted because of the special nature of the arc distances. Finally, section VI gives some comments on further work.

II. THE MODEL

Let t denote the time period index and i be the product number. We define the following quantities for $t=1, 2, \ldots, N$ and $i=1, 2$.

$P_{i,t}(x)$ = The total production cost of producing x units of product i in period t.
$H_{i,t}(x)$ = The inventory holding cost of product i in period t if x is the ending inventory in period t.
$d_{i,t}$ = The known demand for product i in period t.
$I_{i,t}$ = The inventory of product i at the end of period t, before any production in period $t+1$.
$X_{i,t}$ = The production quantity of product i in period t.

We assume that $P_{i,t}(\cdot)$ and $H_{i,t}(\cdot)$ are nonnegative concave functions for all i and t with $P_{i,t}(0)=H_{i,t}(0)=0$, for all i, t. The problem is to minimize $G(X, I)=\sum_i \sum_t [P_{i,t}(X_{i,t})+H_{i,t}(I_{i,t})]$,

Subject to

$$I_{i,t}+I_{i,t,-1}+X_{i,t}-d_{i,t} \qquad i=1, 2; \qquad t=1, 2, \ldots, N$$

$$X_{i,t}X_{2,t}=0 \qquad t=1, 2, 3, \ldots, N$$

$$I_{i,t} \geq 0 \text{ and } X_{i,t} \geq 0 \qquad \text{for } i=1, 2; \qquad t=1, 2, \ldots, N,$$

with initial conditions $I_{i,0}=0$, $I_{2,0}=d_{2,1}$. These initial conditions insure that the problem is feasible and that there will be production of product 1 in the first period. If $I_{2,0}>d_{2,1}$, there may be a period where production occurs starting with nonzero inventory. With no loss of generality we can simply reduce demand $d_{2,2}$ by the amount carried forward to make $I_{2,0}=d_{2,1}$.

Since the objective function is concave, nonlinear programming methods will not reliably provide a solution to this problem. It should be noted that it is possible to replace the constraints $X_{1t}\cdot X_{2t}=0$ with integer linear constraints. This can be used to provide a framework for the proof of Theorem 1, which allows a dynamic programming formulation.

III. CHARACTERISTICS OF OPTIMAL SOLUTIONS

We now present a result which allows us to restrict the search for an optimal schedule to a certain subset of all possible policies. If it were not for the constraint $X_{1,t}, X_{2,t}=0$ we would reference Zangwill [6] for the proof of Theorem 1, for he has proved it without this constraint. Notice that without this constraint, the problem is simply two single product problems which have no interaction and can be solved separately.

THEOREM 1: There exists an optimal, solution (X^*, I^*) such that $X^*_{i,t}I^*_{i,t-1} = 0$ for all $i = 1, 2$; $t = 1, 2, \ldots, N$.

PROOF: Consider any solution (X, I) such that $X_{i,t}I_{i,t-1} > 0$ for some i and t. We show the existence of a solution without this property which has a total cost at least as small as solution (X, I). Let

$$k = \min \{t \mid X_{i,t}I_{i,t-1} > 0, \qquad i = 1 \text{ or } 2\}$$

$$j = \{i \mid X_{i,k}I_{i,k-1} > 0\}.$$

Note that j is well defined since only one product could be produced in period k. Define t^* to be the time of the last previous positive production of product j. (Notice that this must exist or there would not be positive inventory at period k.) The amount produced at t^*, X_{j,t^*}, must be greater than or equal to $I_{j,k-1}$, since $I_{j,t^*-1} = 0$. Consider now two alternative policies (X^A, I^A) and (X^B, I^B). Policy A is defined by subtracting the inventory at the end of period $k-1$ from the production at t^* and adding this amount to production at k. Policy B adds the production of period k to that of period t^* and reduces that of period k to zero. Figure 1 shows a production schedule not satisfying Theorem 1 and the corresponding modifications for Policy A and Policy B.

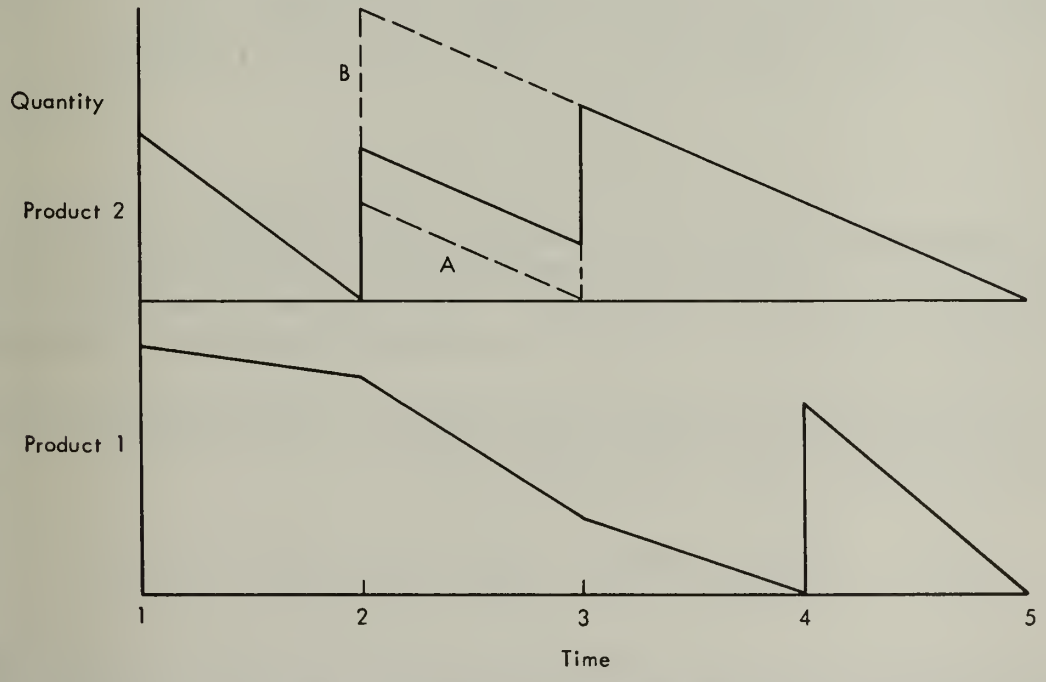

FIGURE 1. A production schedule (X, I) with $X_{2,3}\,I_{2,2}\,0$, which shows policy A and policy B.

Formally, we have:

Policy A $$X^A_{j,t^*} = X_{j,t^*} - I_{j,k-1}$$

$$X^A_j = X_{j,k} + I_{j,k-1}$$

$$I^A_{j,t}=I_{j,t}-I_{j,k-1} \qquad t=t^*,\ t^*+1,\ \ldots,\ k-1.$$

$$X^A_{i,t}=X_{i,t} \qquad \text{all other } i,\ t.$$

$$I^A_{i,t}=I_{i,t} \qquad \text{all other } i,\ t.$$

Policy B

$$X^B_{j,t^*}=X_{j,t^*}+X_{j,k}$$

$$X^B_{j,k}=0$$

$$I^B_{j,t}=I_{j,t}+X_{j,k} \qquad t=t^*,\ t^*+1,\ \ldots,\ k-1.$$

$$X^b_{i,t}=X_{i,t} \qquad \text{all other } i,\ t.$$

$$I^B_{i,t}=I_{i,t} \qquad \text{all other } i,\ t.$$

Both policy A and policy B are feasible and neither one has the property that $X_{i,t}I_{i,t-1}>0$ for any $t\leq k;\ i=1,\ 2$. It is easily verified that $(X, I)=\alpha(X^A, I^A)+(1-\alpha)(X^B, I^B)$ where

$$\alpha=X_{j,k}/(X_{j,k}+I_{j,k-1}).$$

Since the objective function $G(X, I)$ is concave and $0<\alpha<1$, we have $G(X, I)\geq \alpha G(X^A, I^A)+(1-\alpha)$ $G(X^B, I^B)\geq \text{Min}\{G(X^A, I^A), G(X^B, I^B)\}$. Therefore, either policy A or policy B gives as small an objective function value as the value of policy (X, I). Choosing the better of these policies, we have $X_{i,t}I_{k,t-1}=0$ for $i=1, 2;\ t=1, 2, \ldots, k$. This procedure can be repeated if necessary for higher values of t until there are no $X_{i,t}I_{i,t-1}>0$.

LEMMA 1: By restricting ourselves to policies having $X^*_{i,t}I_{k,t-1}=0$, the optimal order $X^*_{i,t}$ placed at any period for product i is one of a finite number of values.

$$X^*_{i,t}=0,\quad d_{i,t},\ \ldots,\quad \sum_{z=t}^{q} d_{i,z} \qquad t<q\leq N.$$

Also, the optimal inventory $I^*_{i,t}$, at the end of any period, t, is equal to one of a finite number of values.

$$I^*_{i,t}=0,\quad d_{i,t+1},\ \ldots,\quad \sum_{z=t+1}^{q} d_{i,z} \qquad t+1<q\leq N.$$

PROOF: If either of these statements is not true the result of Theorem 1 must be violated to insure $I^*_{i,t}\geq 0$. Since $P_{i,t}(\cdot)$ and $H_{i,t}(\cdot)$ are both nonnegative we never need consider producing or holding more inventory than required through period N.

IV. A DYNAMIC PROGRAMMING FORMULATION

The results of the previous section provide the means to define a reasonably tractable dynamic programming formulation. We define—

$F_k(0, \gamma)$ = The minimum total cost of periods $k, k+1, \ldots, N$ if the beginning inventory, before production, will last until the beginning of period γ for product 2 and is zero for product 1. $(k+1 \leq \gamma \leq N+1.)$

$F_k(\gamma, 0)$ = The minimum total cost of periods $k, k+1, \ldots, N$ if the beginning inventory, before production, will last until the beginning of period γ for product 1 and is zero for product 2. $(k+1 \leq \gamma \leq N+1.)$

The recursive relations are:

$$F_N(0, N+1) = P_{1,N}(d_{1,N})$$

and

$$F_N(N+1, 0) = P_{2,N}(d_{2,N}).$$

For $\gamma < N+1$, $F_k(0, \gamma) = \min(D, E)$, where

(1) $$D = \min_{k+1 \leq z < \gamma} \left\{ P_{1,k} \sum_{j=k}^{z-1} d_{1,j} + \sum_{j=k}^{z-2} H_{1,j} \sum_{r=j+1}^{z-1} d_{1,r} + \sum_{j=k}^{z-1} H_{2,j} \sum_{r=j+1}^{\gamma-1} d_{2,r} + F_z(0, \gamma) \right\}$$

and

$$E = \min_{\gamma < z \leq N+1} \left\{ P_{1,k} \sum_{j=1}^{z-1} d_{1,j} + \sum_{j=k}^{\gamma-1} H_{1,j} \sum_{r=j+1}^{z-1} d_{1,r} + \sum_{j=k}^{\gamma-2} H_{2,j} \sum_{r=j+1}^{\gamma-1} d_{2,r} + F_\gamma(z, 0) \right\}.$$

For $\gamma < N+1$, $F_k(\gamma, 0) = \min(K, L)$, where

$$K = \min_{k+1 \leq z < \gamma} \left\{ P_{2,k} \sum_{j=k}^{z-1} d_{2,j} + \sum_{j=k}^{z-2} H_{2,j} \sum_{r=j+1}^{z-1} d_{2,r} + \sum_{j=k}^{z-1} H_{1,j} \sum_{r=j+1}^{\gamma-1} d_{1,r} + F_z(\gamma, 0) \right\}$$

and

(2) $$L = \min_{\gamma < z \leq N+1} \left\{ P_{2,k} \sum_{j=1}^{z-1} d_{2,j} + \sum_{j=k}^{\gamma-1} H_{2,j} \sum_{r=j+1}^{z-1} d_{2,r} + \sum_{j=k}^{\gamma-2} H_{1,j} \sum_{r=j+1}^{\gamma-1} d_{1,r} + F_\gamma(0, z) \right\}.$$

Recursive relation (1) is derived in the following way. The decision, z, is the period number starting immediately after the production of product 1 is exhausted. If this period, z, is less than γ we will be in a position as before with no inventory of product 1 and a certain number of periods supply on hand for product 2. If the production of product 1 takes us beyond the period at which product 2 runs out, γ, we have a starting point with no inventory of product 2 and a certain number of periods supply of product 1. Relation (2) is found in a similar manner.

It is assumed in the above that all sums are zero if the upper index limit is less than the initial index. For the special case $\gamma = k+1$, expressions D and K must be set equal to $+\infty$, since we can not

produce for only one period. For $\gamma = N+1$ another special case arises which can be handled by a notational convention. The total number of $F_k(\cdot, \cdot)$ values which must be evaluated is $2\sum_{i=1}^{N} i = N(N-1)$. with a slight reduction if initial conditions are specified. The number of calculations of expressions of the type on the right side of (1) and (2) is approximately $N(N+1)(2N+1)/3$.

V. A SHORTEST ROUTE NETWORK FORMULATION

In an effort to reduce the computational effort required to solve this problem for reasonably large values of N it was discovered that the problem can be formulated as a shortest route network model. The arc length definitions have some regularity that allow computational reduction in the shortest route algorithm of Minty, [3], [4]. We define the network as (η, A) where

η = the set of nodes
A = the set of arcs.

Let $\eta = [(i, j) \mid i = 0, 1, 2 \ldots, N; j = 0, 1, 2, \ldots, N]$. Arcs are defined in the following way. From a node (i, j) with $i < j < N$ there exists a directed arc to node (t, j) for all $i < t \leq N$, except node (j, j). No other arcs originate from arc (i, j) $(i < j)$. From a node (i, j) with $N > i > j$ there exists a directed arc to node (i, t) for all $j < t \leq N$, except node (i, i). There are no other arcs originating at these nodes. From nodes (i, N) with $i < N$ directed arcs exist to all arcs (t, N) $i < t$, including node (N, N). From nodes (N, j) directed arcs exist to nodes (N, t) for $j < t$), including node (N, N). Notice that even though, for convenience, we defined nodes (i, i) for $0 < i < N$ they are not used in any way and can be neglected.

The above definitions do not help the intuition. The idea is the following. A node, say (i, j), $i \neq j$, represents the event: running out of inventory of product 1 at the end of period i and running out of product 2 at the end of period j. Movement on an arc represents a production decision on the amount of product to produce. This decision is always made for the produce which has the smallest period subscript, that is the produce that is exhausted first. For example if $N = 10$ and we are at node (3, 7), traversing the directed arc to node (5, 7) represents the decision to produce 2 periods demand for product 1. Notice that node (7, 7) is not allowed as a destination for it would require production of both products. This supply of product 1 will be exhausted at the end of period 5. Since (5, 7) has $5 < 7$ the next decision is also made for product 1. If, for instance, we then move to node (9, 7) the next decision is made for product 2, since it is now exhausted at the end of an earlier period, 7, than product 1, which is exhausted at the end of period 9.

The reader will notice the similarity of this shortest route formulation and the shortest route network for a single product model.

The arc lengths are defined as the total production and holding costs incurred using the above correspondences. Let $A((i, j), (k, j))$ be the distance between node (i, j) for $i < j < N$ and node (k, j), $i < k \neq j$,

$$A((i, j), (k, j)) = P_{1, i+1} \sum_{n=i+1}^{k} d_{1, n} + \sum_{n=i+1}^{k} H_{1, n} \sum_{r=n+1}^{k} d_{1, r}. \tag{3}$$

If $j = N$, then distance is defined for $k = N$ by using the same formula. From node (i, j) with $j < i$ the

directed arc to node (i, k) with $j < k \neq i$ has length.

$$A((i,j),(i,k)) = P_{2,j+1} \sum_{n=j+1}^{k} d_{2,n} + \sum_{n=j+1}^{k-1} H_{2,n} \sum_{r=n+1}^{k} d_{2,r}. \tag{4}$$

If $i = N$, the distance is defined for $k = N$ by using (4). In (3) and (4) a convention that $\sum_{j=M}^{Q} H_{i,j}(\cdot) = 0$ if $Q < M$ is used.

The shortest route problem is to start at that node representing the initial conditions and travel to node (N, N) incurring the minimum possible total distance traveled. A small example will more fully illustrate the correspondence of a production schedule with a path through the network. Let $N = 9$ and suppose the initial inventories at the beginning of period 1 (end of period 0) are $I_{1,0} = 0$, $I_{2,0} = d_{2,1}$. Product 2 will be depleted at the end of period 1. Thus the starting node is (0, 1). Suppose the node sequence followed in a path is (0, 1), (2, 1), (2, 4), (5, 4), (5, 7), (6, 7), (9, 7), and (9, 9) as indicated in Figure 2. The corresponding production schedule is shown in Figure 3. This schematic shows when each product is exhausted and consequently the times of production. It is easily verified, given a feasible production schedule as is the above schematic, that this schedule corresponds to a unique path through the network and that total arc length agrees with the total production and holding costs.

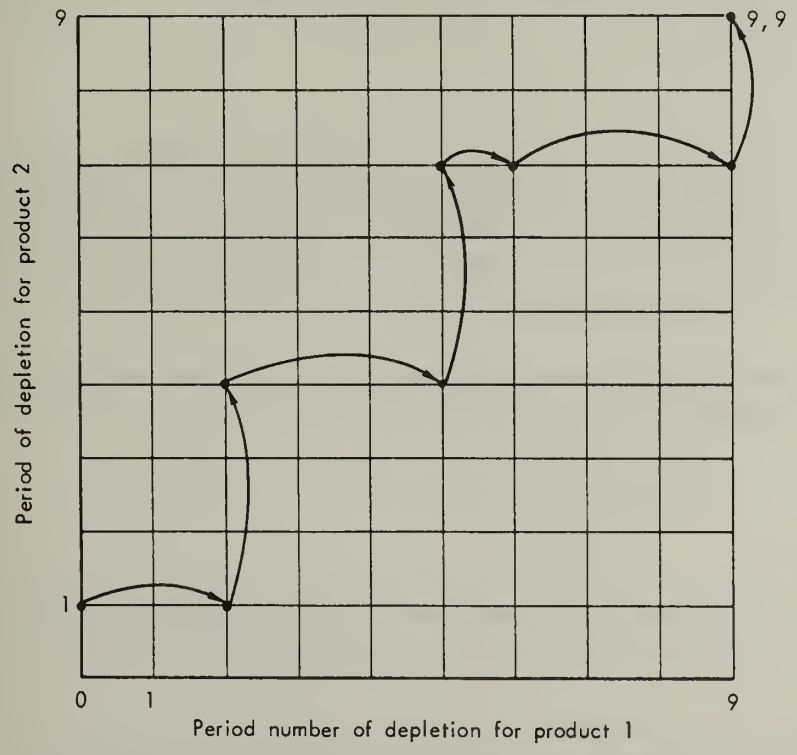

FIGURE 2. Network representation of a production schedule.

This network representation has $(N+1)^2$ nodes and $N(N+1)(2N+1)/3$ arcs. The dynamic programming approach given in section IV is equivalent to determining the shortest path from all nodes to node (N, N). It is not always necessary to perform all these calculations if one uses a shortest

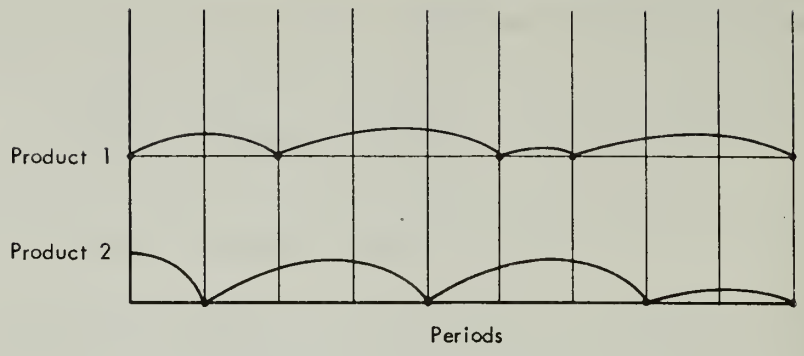

FIGURE 3. A production schedule.

path algorithm such as Minty's [3], [4], which ends at the first time node (N, N) is reached. This could allow a reduction in computing time. Notice that to apply Minty's algorithm to this problem, it is not necessary to calculate all arc length before initiating the algorithm. The only arc lengths needed are those which emanate from a permanently labeled node, to the next closest unlabeled node. This information can be maintained easily due to the special structure of the network, and arc lengths can be generated as needed. Since the total number of arcs is the same as the number of expressions on the right of (1) and (2) that must be evaluated by use of the dynamic programming solution, if some arc distances are not required, the shortest route formulation would be more computationally efficient than dynamic programming.

VI. COMMENTS

Results of a "planning horizon" nature are not possible with the general concave cost functions we have used. If one defined particular functions, such as fixed set up cost plus linear marginal production cost, it may be possible to derive planning horizon results simular to Wagner and Whiten [5] and Eppen, Gould, and Pashigan [1]. It is conceptually more difficult because of the fact that the products must not be allowed to run out simultaneously, thereby making the definition of a regeneration point more difficult.

The shortest route network model can be used to solve problems with more than two products. Suppose there are k products and N periods. Define the node set, η, of size $(N+1)^k$ as

$$\eta = \{(i_1, i_2, \ldots, i_k) \mid i_j \epsilon \{0, 1, 2, \ldots, N\}, \quad \forall_j\}.$$

Then a node $(i_1, i_2, \ldots, i_k)$ with $i_j \neq i_m$ for all $j \neq m$ corresponds to running out of product q at the end of period i_q. Movement from a node would be allowed only in the direction of increasing the smallest of the i_q subscripts. Arc lengths can be defined simularly to formulae (3) and (4) for the two product model.

It would be satisfying to report that an efficient algorithm has been discovered to solve the model of this paper. At the present time the best that can be said is that the use of Minty's algorithm can be somewhat simplified in terms of computation by calculating only those arc lengths that are required. It is hoped that viewing this model in the framework of a special structure shortest route problem will lead to further algorithmic research on problems of this type. It is a problem worthy of research for it is very seldom that a production-inventory situation can be viewed as a single product decision problem.

This model is a first step in trying to attack the deterministic multiproduct inventory problem, when constraints on joint production of an either-or type are present. Extension or out results to the multi-product case are immediate with minor notational difficulties.

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TWO MACHINE FLOW SHOP SCHEDULING PROBLEMS WITH SEQUENCE DEPENDENT SETUP TIMES: A DYNAMIC PROGRAMMING APPROACH*

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ABSTRACT

This paper considers the two different flow shop scheduling problems that arise when, in a two machine problem, one machine is characterized by sequence dependent setup times. The objective is to determine a schedule that minimizes makespan. After establishing the optimality of permutation schedules for both of these problems, an efficient dynamic programming formulation is developed for each of them. Each of these formulations is shown to be comparable, from a computational standpoint, to the corresponding formulation of the traveling salesman problem. Then, the relative merits of the dynamic programming and branch and bound approaches to these two scheduling problems are discussed.

1. INTRODUCTION

Most results offered to date for the flow shop scheduling problem have required the simplifying assumption that setup times on each machine are independent of job sequence. Notable exceptions to this rule have been the recent branch and bound algorithms for the generalized scheduling problem that are based on its graph-theoretic representation [2], [7], [10]. This paper is concerned with the two different scheduling problems that result from the classic two machine scheduling problem when one machine is assumed to have setup times that are dependent on job sequence. For both problems, the objective is to determine a job sequence for each machine (schedule) that minimizes the total elapsed time to complete all jobs (makespan). A minimal makespan schedule will be hereafter referred to as an optimal schedule.

Attention is focused on these two particular problems because they are closely related to both the classic two machine scheduling problem and the traveling salesman problem (TSP). Therefore, there is a large potential carryover of both solution properties and solution methods from these latter problems to the former ones. In particular, Johnson [13] has shown that the class of schedules for which the job sequence on each machine is identical (permutation schedules) contains an optimal schedule for the classic two machine problem. Also, Bellman [3] and Held and Karp [11] have developed essentially equivalent dynamic programming (DP) formulations for the TSP, or equivalently the single machine scheduling problem with sequence dependent setup times. These formulations have been shown to

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be well-suited for handling relatively small problems (14 or fewer jobs). These considerations suggest investigating for each of these problems both the optimality of permutation schedules and the suitability of DP for finding such a schedule.

The developments presented here have considerable practical significance. Single machine scheduling models have hitherto been employed as approximations in situations where a two or more facility model would have been more appropriate. One example is provided by the scheduling problem that arises in conjunction with paper production.* The objective for this problem is to determine a schedule that minimizes makespan subject to meeting certain due date constraints. The production system involved here consists of essentially two machines. The first machine (batch digestor) prepares the stock or slurry for the second machine (paper machine) which forms the finished sheets. Significant setup times that depend on the sequence in which jobs are processed are incurred on the second machine. This particular system has been modeled by Pierce and Hatfield [16] as a single machine characterized by sequence dependent setup times by ignoring the first machine. However, in most cases it is not sufficient to consider only the paper machine in determining an optimal schedule. A schedule that is optimal for the paper machine alone may be decidedly nonoptimal for the two machine system because of the intermittent, rather than simultaneous, arrival of jobs at the paper machine. Thus, one of the two distinct two machine models considered in this paper is more appropriate for this situation than the single machine model of [16]. The due date constraints considered in [16] can readily be incorporated into the DP formulation of this two machine model.

This paper essentially consists of three separate parts. In section 2, the two scheduling problems considered here are concisely defined, and the optimality of permutation schedules for each of them is established. A DP formulation for each of these problems is developed in section 3. The computational requirements of each formulation are shown to be comparable to those of the DP formulation of the TSP. In section 4, the relative merits of the DP and branch and bound (B&B) approaches to each of these problems for various problem sizes are fully discussed. The DP approach is shown to have a distinct advantage in both cases for problems of small size.

2. PROBLEM DEFINITION

The set of assumptions which define the classic two machine scheduling problem are well documented in Johnson [13]. This paper considers the two variants of this problem that result when the assumption of sequence independent setup times is relaxed by allowing setup times on one machine to be sequence dependent. A machine on which setup times are sequence dependent will be referred to as a type D machine while a machine on which setup times are sequence independent will be referred to as a type I machine. By use of this notation, the two problems considered here will be separately referred to as problem type ID and problem type DI.

For a type D machine, the combined setup and processing time of job j if it immediately follows job i will be denoted by s_{ij}, $i=ps, 1, 2, \ldots, n, j=1, 2, \ldots, n$. Note that an imaginary $(n+1)$st job is added to represent the preliminary state (ps) of the machine. It is assumed that a return to ps is not required upon completion of the n jobs. For a type I machine, the combined setup and processing time of job j will be denoted by p_j, $j=1, 2, \ldots, n$.

As noted previously, the objective for each problem type is to determine an optimal (minimal makespan) schedule. The class of permutation schedules contains an optimal schedule for any scheduling

*The authors are indebited to Charles F. Long, Bethlehem Steel Corporation, for bringing this example to their attention.

problem of either type ID or type DI. The proofs of these results can be found in [9]. The proof of the former (latter) result is similar to that of Conway, Maxwell, Miller [8] of the existence of an optimal schedule for problems involving m type I machines for which the job sequences on the first (last) two machines are the same. Each proof essentially demonstrates that, given an arbitrary job sequence on each machine, the job sequence on the type I machine can be made the same as that on the type D machine without increasing the makespan.

3. DYNAMIC PROGRAMMING FORMULATIONS

A concept of sequence dominance, distinct from that employed in [12] and [17] by virtue of the presence of sequence dependent setup times, can be defined for the two problems considered here. The associated set of sequence dominance conditions provides the basis for a forward recursion DP formulation of problem type ID. This formulation is a direct extension of the DP formulation of the TSP. A DP formulation of problem type DI analagous to that of problem type ID does not satisfy the set of sequence dominance conditions and hence is not valid. However, a backward recursion DP formulation, employing a single state variable to indicate the availability time of the second machine can be developed for problem type DI. This formulation possesses two useful properties which greatly reduce its computational requirements.

3.1 A Dynamic Programming Formulation of Problem Type ID

A permutation schedule is completely described by a particular permutation of the job numbers $1, 2, \ldots, n$. Any such permutation will be referred to as a full sequence. Consider a sequence of $r(1 \leq r \leq n)$ jobs that corresponds to the first r positions of a full sequence. Any such sequence will be referred to as a presequence of size r. The time required to complete a presequence, i.e., the time between its start on machine 1 (schedule time zero) and its completion on machine 2, will be referred to as the presequence completion time.

For any presequence α_r, let $T_m(\alpha_r)$ $(m=1, 2)$ be the time that the job in position r of α_r is completed on machine m. Let α_r' and α_r'' be two different presequences of r jobs that contain the same r jobs and end with the same job. If

$$T_1(\alpha_r') \leq T_1(\alpha_r'') \tag{3.1.1}$$

and

$$T_2(\alpha_r') \leq T_2(\alpha_r''), \tag{3.1.2}$$

then α_r' is said to dominate α_r''. Let π be any permutation of the $(n-r)$ jobs not contained in α_r' and α_r''. Conditions (3.1.1) and (3.1.2) insure that the full sequence $(\alpha_r''\pi)$ cannot have a smaller makespan than the full sequence $(\alpha_r'\pi)$. Thus, all full sequences starting with α_r'' can be eliminated from further consideration. Conditions (3.1.1) and (3.1.2) are the sequence dominance conditions for problem types ID and DI.

Consider problem type ID. Since machine 1 is of type I, (3.1.1) must hold as an equality. Thus, α_r' dominates α_r'' if the completion time on machine 2 (flow time) of the job in position r under α_r' is less than equal that under α_r''. This result forms the basis for a forward recursion DP formulation in the presequence completion time for problem type ID.

Let J be the set of all n jobs, j_l be a single job, and $J_k=\{j_1, j_2, \ldots, j_k\}$ be a subset of k $(0 \leq k \leq n-1)$ jobs from $\{J-j_l\}$. Suppose that the first $(k+1)$ jobs of some optimal full sequence have been completed, with the remaining $(n-k-1)$ jobs yet to be done. Regard the first k jobs as some set of jobs J_k, and the $(k+1)$st job as some job j_l. For any J_k and j_l, define $f(ps; J_k; j_l)$ to be the minimum time required to complete the k jobs of J_k (in some order) and then job j_l, where machine 2 starts from the preliminary state, ps. Then $f(ps; J_k; j_l)$ is given by

$$(3.1.3) \qquad f(ps; J_k; j_l) = \min_{j_p \in J_k} \left[f(ps; \{J_k - j_p\}; j_p) + I(j_l | \rho^p_{k-1} j_p) + s_{j_p j_l} \right] \qquad \text{for } k=1, 2, \ldots, n-1,$$

where ρ^p_{k-1} is the optimal presequence of the $(k-1)$ jobs of $\{J_k - j_p\}$ and $I(j_l | \rho^p_{k-1} j_p)$ is the idle time incurred on machine 2 immediately preceding job j_l when the presequence $(\rho^p_{k-1} j_p)$ is used. Note that $I(j_l | \rho^p_{k-1} j_p)$ is given by

$$I(j_l | \rho^p_{k-1} j_p) = \max \left[\sum_{j \in \{J_k, j_l\}} p_j - f(ps; \{J_k - j_p\}; j_p), 0 \right].$$

Let j_m be a choice of j_p that corresponds to the minimum in (3.1.3). If $(\rho^*_k j_l)$ is an optimal presequence associated with $f(ps; J_k; j_l)$, then $(\rho^*_k j_l)$ is given by $(\rho^m_{k-1} j_m j_l)$. The recursive procedure of (3.1.3) is initiated by use of the known relation

$$f(ps; J_0; j_l) = p_{j_l} + s_{ps j_l} \qquad \text{for } k=0,$$

where $J_0 = \phi$, the null set. Once $f(ps; J_{n-1}; j_l)$ and an associated optimal full sequence $(\rho^*_{n-1} j_l)$ have been obtained for each of the n possible jobs j_l, a solution to the problem is given by

$$\min_{j_l \in J} f(ps; J_{n-1}; j_l).$$

Examination of relation (3.1.3) reveals that it establishes $f(ps; J_k; j_l)$ by use of the sequence dominance conditions (3.1.1) and (3.1.2). For each $j_p \in J_k$,

$$f(ps; \{J_k - j_p\}; j_p) + I(j_l | \rho^p_{k-1} j_p) + s_{j_p j_l} = T_2(\rho^p_{k-1} j_p j_l).$$

Thus, the minimization in (3.1.3) selects a presequence $(\rho^m_{k-1} j_m j_l)$ from the k presequences $(\rho^p_{k-1} j_p j_l)$ that dominates the remaining $(k-1)$ presequences. Since the relations $T_1(\rho^m_{k-1} j_m j_l) = T_1(\rho^p_{k-1} j_p j_l)$ and $T_2(\rho^m_{k-1} j_m j_l) \leq T_2(\rho^p_{k-1} j_p j_l)$ must hold for each of the $(k-1)$ presequences $(\rho^p_{k-1} j_p j_l)$ corresponding to a job $j_p \in \{J_k - j_m\}$, each of these presequences is dominated by the presequence $(\rho^m_{k-1} j_m j_l)$ and therefore can be eliminated from further consideration.

It is instructive to compare the computational requirements (storage and time) of this DP formulation of an n job problem of type ID to those of the DP formulation of an $(n+1)$ city TSP. It can be readily seen that the calculation of $I(j_l | \rho^p_{k-1} j_p)$ required for (3.1.3) is essentially the only computational difference between these two formulations. The computer storage requirements of each problem

solved at stage k ($0 \leq k \leq n-1$), and hence at stage k itself, are identical for both formulations. For each problem solved at stage k, $I(j_l | \rho^p_{k-1} j_p)$ must be calculated for each of the k alternatives compared. Thus, the computation time requirements of the former formulation are slightly greater than those of the latter.

3.2 A Dynamic Programming Formulation of Problem Type DI

A forward recursion DP formulation in the presequence completion time would not be valid for problem type DI. Since machine 1 is of type D, (3.1.1) does not necessarily hold as an equality. Thus, α'_r may or may not dominate α''_r if the flow time of the job in position r under α'_r is less than or equal that under α''_r. Assume $f(ps; J_k; j_l)$ is defined for problem type DI as it was before for problem type ID, where now machine 1 starts from the preliminary state. A forward recursion relation analagous to (3.1.3) would not be valid because $f(ps; J_k; j_l)$ cannot, in general, be expressed in terms of the minimum completion time $f(ps; \{J_k - j_p\}; j_p)$ for some job $j_p \epsilon J_k$ and the idle time and job time that result on machine 2 from appending job j_l to the end of the associated presequence $(\rho^p_{k-1} j_p)$.

As noted previously, a backward recursion DP formulation can be developed for problem type DI. Let J and j_l be defined as before. Assume initially that the search for an optimal full sequence is to be restricted to those full sequences that have job j_l in sequence position n. The DP formulation of the problem of finding an optimal full sequence ending with job j_l will be referred to as the DP formulation of problem type DI with specified last job j_l. Consider a sequence of r jobs ($1 \leq r \leq n$) that corresponds to the last r positions of a full sequence. Any such sequence will be referred to as a postsequence of size r. The time required to complete a postsequence, i.e., the time between its start on machine 1 and its completion on machine 2, given the identity of the job immediately preceding it and the time at which machine 2 is available to start it will be referred to as the postsequence completion time. The DP formulation of problem type DI with specified last job j_l utilizes a backward recursion in the postsequence completion time.

Let j_i be a single job different from j_l, and $J_k = \{j_1, j_2, \ldots, j_k\}$ be a subset of k ($0 \leq k \leq n-2$) jobs from $\{J - j_l - j_i\}$. For $k = n-1$, let $j_i = ps$ and $J_{n-1} = \{J - j_l\}$. Consider the last $(k+1)$ jobs of some optimal full sequence with a specified last job. Regard this last job as some job j_l, and the remaining k jobs as some set of jobs J_k. Let the immediate predecessor of these $(k+1)$ jobs be job j_i, which will be referred to as the preliminary job. Define the integer valued state variable L to be the length of time between the completion of the preliminary job j_i on machine 1 and its completion on machine 2. For a specified j_l, and any J_k and j_i, define $f(j_i; J_k; j_l \mid L)$ to be the minimum time required to complete the jobs of J_k (in some order) and then job j_l, with job j_i as the preliminary job, given the value of L. Note that $L \geq p_{j_i}$ for $k = 0, 1, \ldots, n-3$ and $L = p_{j_i}$ for $k = n-2$; L is defined to be zero for $k = n-1$. Then $f(j_i; J_k; j_l | L)$ is given by

$$(3.2.1) \qquad f(j_i; J_k; j_l | L) = \min_{j_p \epsilon J_k} [s_{j_i j_p} + f(j_p; \{J_k - j_p\}; j_l | L'(j_p))] \qquad \text{for } k = 1, 2, \ldots, n-1,$$

where

$$L'(j_p) = \max (L - s_{j_i j_p}, 0) + p_{j_p}.$$

The state variable $L'(j_p)$ is the length of time between the completion of job j_p on machine 1 and its completion on machine 2 if job j_p immediately follows the preliminary job j_i. The recursive procedure

of (3.2.1) can be considerably simplified by the two properties of $f(j_i; J_k; j_l|L)$ established by the following theorem.

THEOREM 1: Consider $f(j_i; J_k; j_l|L)$ for k=0, 1, . . ., $n-3$. If $(\rho_k^* j_l)$ is an optimal postsequence associated with $f(j_i; J_k; j_l|p_{j_i})$, then an optimal postsequence associated with $f(j_i; J_k; j_l|L)$ for any $L \geq p_{j_i}$ is given by $(\rho_k^* j_l)$. Furthermore,

$$f(j_i; J_k; j_l|L) = \begin{cases} f(j_i; J_k; j_l|P_{j_i}) & \text{if } p_{j_i} \leq L \leq [f(j_i; J_k; j_l|p_{j_i}) - C] \\ L + C & \text{if } L > [f(j_i; J_k; j_l|p_{j_i}) - C], \end{cases} \tag{3.2.2}$$

where $C = \sum_{j \in \{J_k, j_l\}} p_j$.

The proof of Theorem 1 is given in the appendix. Due to Theorem 1, only $f(j_i; J_k; j_l|p_{j_i})$ and an associated optimal postsequence $(\rho_k^* j_l)$ must be obtained. Therefore, the recursive procedure of (3.2.1) can be replaced by

$$f(j_i; J_k; j_l|p_{j_i}) = \min_{j_p \in J_k} [s_{j_i j_p} + f(j_p; \{J_k - j_p\}; j_l|L'(j_p))], \qquad \text{for } k = 1, 2, \ldots, n-1, \tag{3.2.3}$$

where

$$L'(j_p) = \max(p_{j_i} - s_{j_i j_p}, 0) + p_{j_p}.$$

Note that (3.2.2) provides a simple means of evaluating $f(j_p; \{J_k - j_p\}; j_l \mid L'(j_p))$ for any $L'(j_p) > p_{j_p}$. Let $(\rho_{k-1}^p j_l)$ be the optimal postsequence of the $(k-1)$ jobs of $\{J_k - j_p\}$ and job j_l associated with $f(j_p; \{J_k - j_p\}; j_l|p_{j_p})$, and let j_m be a choice of j_p that corresponds to the minimum in (3.2.3). An optimal postsequence $(\rho_k^* j_l)$ associated with $f(j_i; J_k; j_l|p_{j_i})$, and hence with $f(j_i; J_k; j_l|L)$ for any $L \geq p_{j_i}$, is then given by $(j_m \rho_{k-1}^m j_l)$. The recursive procedure of (3.2.3) is initiated by use of the known relation

$$f(j_i; J_0; j_l|p_{j_i}) = \max(s_{j_i j_l}, p_{j_i}) + p_{j_l} \qquad \text{for } k = 0.$$

The case $k = n - 1$ in (3.2.3) gives an optimal full sequence with specified last job j_l. This completes the DP formulation of problem type DI with specified last job j_l.

A DP formulation of problem type DI is obtained from the above formulation by successively designating each of the n jobs to be the last job j_l. Once $f(ps; J_{n-1}; j_l|0)$ and an associated optimal full sequence $(\rho_{n-1}^* j_l)$ have been obtained for each of the n possible jobs j_l, a solution to the problem is given by

$$\min_{j_l \in J} f(ps; J_{n-1}; j_l|0).$$

Once again, it is instructive to compare the computational requirements of this DP formulation of an n job problem of type DI to those of the DP formulation of an n city TSP. Consider the computational requirements of the DP formulation of an n job problem of type DI with specified last job j_l and those of the DP formulation of an n city TSP. Because Theorem 1 renders separate consideration of different values of L unnecessary, there is a strong computational similarity between these two for-

mulations. The computer storage requirements of each problem solved at stage $k(0 \leq k \leq n-1)$, and hence at stage k itself, are identical for both formulations. For each problem solved at stage k, $L'(j_p)$ must be calculated for each of the k alternatives compared, and (3.2.2) must be used to determine $f(j_p; \{J_k - j_p\}; j_l | L'(j_p))$ for any of the k alternatives for which $L'(j_p) > p_{j_p}$. Thus, the computation time requirements of the former formulation are slightly greater than those of the latter. Consequently, since this DP formulation of an n job problem of type DI requires that all n possible last jobs j_l be considered, its computation time requirements are on the order of n times as large as those of the DP formulation of an n city TSP. The computer storage requirements of both formulations are, however, the same.

4. DISCUSSION OF THE DYNAMIC PROGRAMMING APPROACH

The computational limitations for the DP formulation of the TSP, and thus for the DP formulations of problem types ID and DI, are primarily imposed by computer storage considerations rather than by computation time considerations. The limits on problem size n for the DP formulation of the TSP have received some attention [3], [5], [9], [11]. Bellmore and Nemhauser [5] indicate that with the use of auxiliary storage, the largest problem solvable on a computer with a $32k$ core is $n = 15$. Therefore, under these same assumptions, the largest problems of type ID and type DI solvable are approximately $n = 14$ and $n = 15$, respectively.

Several B&B approaches to machine scheduling problems reported in the literature can, with appropriate modifications, be employed to solve problems of types ID and DI. Some discussion of these approaches and comparison of them with the DP approach of this paper is appropriate. Charlton and Death [7] and Balas [2] have reported B&B algorithms for the generalized machine scheduling problem based on the equivalence of this problem to the problem of finding a minimaxal path in a disjunctive graph. These algorithms can be modified to incorporate the notion of a preliminary state and to restrict consideration to permutation schedules only, both of which are necessary in treating problem types ID and DI. It is important to note, however, that if the notion of a preliminary state is incorporated, the associated disjunctive graph cannot be defined for a problem of either type unless the job in sequence position one is specified. Thus, this modification necessitates that these algorithms consider n different problems, one corresponding to each possible first job, in order to solve an n job problem of either type. Ignall and Schrage [12] and Lomnicki [15] have reported similar B&B algorithms for the classic three machine scheduling problem. Their algorithms are based on building increasingly larger presequences of jobs by successive branching operations, and associating a lower bound with each new presequence created. Corwin [9] has appropriately modified these lower bounds to enable these algorithms to treat problem types ID and DI.

Consider the B&B counterparts to the DP algorithms developed in this paper for problem types ID and DI that are obtained by appropriately modifying the B&B algorithms discussed above. The computer storage requirements of each of these algorithms are both minimal for any problem size n and invariant between different problems of the same size as a result of using the newest active node search procedure in implicitly enumerating all permutation schedules. It is important to note that this search procedure is generally acknowledged to be relatively inefficient with respect to computation time considerations (see [1] and [14]), so that these desirable storage properties are not achieved without some sacrifice. The computation time requirements of each of these algorithms depend on the number of nodes that must be created in order to obtain and validate an optimal schedule. This number can vary considerably between different problems of the same size, and in fact may be quite excessive even for small problem sizes. The computation time requirements of these algorithms thus have the

undesirable property of being problem dependent. It is important to note that this is not the case with the DP algorithms advanced here for problem types ID and DI. Their computational requirements do not vary between different problems of the same size, and can be precisely predicted. Recall the comparisons made previously between the computational requirements of the DP algorithms for problem types ID and DI and those of the DP algorithm for the TSP. Bellmore and Nemhauser [5] recommend DP over B&B for smaller TSP's because of the assurance that no problem of a given size can result in excessive computation time requirements. This same assurance can be given for the DP algorithms developed in this paper for problem types ID and DI, but cannot be given for their B&B counterparts due to the problem dependent nature of their computational requirements.

On the basis of computation time assurances, it can be concluded that the DP algorithm for problem type ID is preferable to its B&B counterparts for problems involving approximately 14 or fewer jobs. This same conclusion must be drawn somewhat more cautiously for the DP algorithm for problem type DI because of the necessity to explicitly consider the different problems that result from each possible specification of a last job. However, as noted previously, the graph-theoretic based B&B counterparts must in an analogous fashion explicitly consider each possible specification of a first job. Thus, on the basis of computation time assurances, the DP algorithm for problem type DI appears to be preferable to these B&B counterparts.

For problems of both types involving larger numbers of jobs, it should be noted that the B&B approach may provide an optimal schedule, while both the B&B and the DP approaches can be employed to give "good" schedules. The B&B algorithms considered here would be capable of solving some particular problems whose size exceeds the known limits on their DP counterpart. Since each of these B&B algorithms produces successively improving schedules, a "good" schedule is generally available in the event the algorithm must be terminated prematurely for a particular problem. Also, it appears that the method of successive approximations can be used to overcome the dimensionality problem inherent in the DP approach to provide "good" schedules for larger problems of both types. It is envisioned that this method would be employed in conjunction with the DP algorithm for each problem type in much the same manner as it is employed by Buzzacott and Dutta [6] in conjunction with a DP algorithm for a variant of the single machine scheduling problem.

APPENDIX

Proof of Theorem 1. For any postsequence $(\rho_K j_l)$ following j_i, let $PT_2(\rho_K j_l|L)$ be the postsequence time given the value of L. Let $(\rho'_k j_l)$ be any postsequence different from all optimal postsequences associated with $f(j_i, J_k; j_l|p_{j_i})$. Note that there can be no postsequence $(\rho'_k j_l)$ different from $(\rho^*_k j_l)$ unless $k \geq 2$. It must be shown that, for $k=2, 3, \ldots, n-3$,

$$PT_2(\rho^*_k j_l|L) \leq PT_2(\rho'_k j_l|L) \qquad \text{for all } L \geq p_{j_i}. \tag{1}$$

Consider the case $L=p_{j_i}$. Note that

$$PT_2(\rho^*_k j_l|p_{j_k}) = f(j_i; J_K; j_l|p_{j_i}).$$

From the definitions of $(\rho^*_k j_l)$ and $(\rho'_k j_l)$,

(2) $$PT_2(\rho_k^* j_l | p_{ji}) < PT_2(\rho_k' j_l | p_{ji}).$$

The postsequence $(\rho_k^* j_l)$ can be shifted to the right on machine 2, leaving an interval of time $[PT_2(\rho_k^* j_l | p_{ji}) - C]$ between the start of the first job of $(\rho_k^* j_l)$ on machine 1 and its start on machine 2. Similarly, the postsequence $(\rho_k' j_l)$ can be shifted to the right on machine 2, leaving an interval of time $[PT_2(\rho_k' j_l | p_{ji}) - C]$ between the start of the first job of $(\rho_k' j_l)$ on machine 1 and its start on machine 2. It follows from the definition of $PT_2(\rho_k^* j_l | p_{ji})$ and from (2) that

$$p_{ji} \leq [PT_2(\rho_k^* j_l | p_{j_i}) - C] < [PT_2(\rho_k' j_l | p_{ji}) - C].$$

In order to establish (1), the following three cases must be considered:

CASE 1. $p_{ji} \leq L \leq [PT_2(\rho_k^* j_l | p_{ji}) - C]$. Here

$$PT_2(\rho_k^* j_l | L) = PT_2(\rho_k^* j_l | p_{j_i})$$

and

$$PT_2(\rho_k' j_l | L) = PT_2(\rho_k' j_l | p_{j_i}).$$

Therefore, $PT_2(\rho_k^* j_l | L) < PT_2(\rho_k' j_l | L)$ by (2).

CASE 2. $[PT_2(\rho_k^* j_l | p_{ji}) - C] < L < [PT_2(\rho_k' j_l | p_{ji}) - C]$. Here

$$PT_2(\rho_k^* j_l | L) = PT_2(\rho_k^* j_l | p_{j_i}) + \{L - [PT_2(\rho_k^* j_l | p_{j_i}) - C]\} = L + C$$

and

$$PT_2(\rho_k' j_l | L) = PT_2(\rho_k' j_l | p_{ji}).$$

Therefore, $PT_2(\rho_k^* j_l | L) < PT_2(\rho_k' j_l | L)$ since $L + C < PT_2(\rho_k' j_l | p_{ji})$.

CASE 3. $[PT_2(\rho_k' j_l | p_{ji}) - C] \leq L$. Here

$$PT_2(\rho_k^* j_l | L) = PT_2(\rho_k^* j_l | p_{ji}) + \{L - [PT_2(\rho_k^* j_l | p_{ji}) - C]\} = L + C$$

and

$$PT_2(\rho_k' j_l | L) = PT_2(\rho_k' j_l | p_{ji}) + \{L - [PT_2(\rho_k' j_l | p_{ji}) - C]\} = L + C.$$

Therefore, $PT_2(\rho_k^* j_l | L) = PT_2(\rho_k' j_l | L)$.

Combining the results of Cases 1, 2, and 3 gives (1). Therefore, $(\rho_k^* j_l)$ is optimal for any $L \geq p_{ji}$.

Since $(\rho_k^* j_l)$ is optimal for any $L \geq p_{ji}$, it follows that

(3) $$f(j_i; J_k; j_l|L) = PT_2(\rho_k^* j_l|L) \qquad \text{for } L \geq p_{ji}.$$

Equation (3.2.2) follows directly from (3) and the results of Cases 1, 2, and 3.

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POLLUTION ABATEMENT AND ECONOMIC GROWTH: AN APPLICATION OF THE VON NEUMANN MODEL OF AN EXPANDING ECONOMY*

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ABSTRACT

The generalized von Neumann model of an expanding economy permits both joint products and alternative processes. Pollutants may be explicitly treated as the normal by-products of certain economic processes and "clean" and "dirty" production alternatives may be simultaneously available. The paper describes some methods for incorporating pollutants in the model and several techniques that may be used to study the relationship between pollution abatement and economic growth. Numerical examples are used to illustrate cases of direct control of processes, consumer demand for "eliminated pollutant" and the limitation of total produced pollutants by exogenous constraint. In the last case an effluent charge is calculated.

INTRODUCTION

Most economic models follow the convention that the significant outputs of economic activity are "goods" and those outputs that are recognized as "bads" are external to the market system and therefore external to the economic models used to study the system. Economists have recognized evils in the prevailing economic systems of their times, but with notable exceptions (for example [1, 9] economists have treated specific physical products that are undesirable as exceptions). Even the most obvious bads are transformed into a kind of negative good. The collection of garbage is a service and a good, but garbage left uncollected is rarely treated explicitly.

Explicit treatment of bads in unconventional, but bads can be incorporated into existing economic models. For example, there seems no reason that the familiar indifference curve analysis cannot be applied in a plane with a bundle of goods on one axis and a bundle of pollutants on the other. The shape of the curves is unfamiliar, but many of the familiar techniques may be applied.

In a recent article [8] Professor Wassily Leontief has shown that the conventional input-output model can be modified to yield useful results about environmental problems when pollutants are introduced as normal and explicit by-products of economic activity. The purpose of this paper is to adapt this concept to the von Neumann model of an expanding economy and to develop some techniques that appear to be useful in the study of pollution abatement and economic growth.

THE GENERALIZED VON NEUMANN MODEL OF AN EXPANDING ECONOMY

The von Neumann model of an expanding economy, as generalized by Kemeny, Morganstern, and

* The views expressed are those of the author and not necessarily those of the Bureau of Economic Analysis.

Thompson [5] is easily adapted to the explicit treatment of pollutants (i.e., bads) as the normal by-products of economic activity. The following is a brief summary of that model with some minor changes in notation to facilitate the inclusion of pollutants among the products produced.

Consider an economy with m processes and n products. Each process is assumed to yield constant returns to scale. Each process may produce multiple products as outputs. There may be several processes that produce the same product.

The available technology is summarized by the m by n matrices A and B:

$A=[a_{ij}]$, where a_{ij} is the quantity of the jth product required as input to operate the ith production process at the unit level.

$B=[b_{ij}]$, where b_{ij} is the quantity of the jth product produced as output when production process i is operated at the unit level of activity.

Each process (i) operates at an intensity $x_i \geq 0$ and the row vector $X=(x_1, x_2, \ldots, x_m)$ describes the activity of the economy. For our purposes only the relative levels of activity of the processes are significant and hence we may choose units for the goods so as to normalize the vector X. Thus $0 \leq x_1 \leq 1$ and $\sum_{i=1}^{m} x_i=1$.

Each good is assigned a price $y_j \geq 0$ and the column vector $Y=$ column $(y_1, y_2, \ldots, y_n)$ describes the prices effective in the economy. Only the relative prices are significant for our purposes and we may choose the unit of money so that $0 \leq y_j \leq 1$ and $\sum_{j=1}^{m} y_j=1$.

The row vector XA is the complete list of all inputs required for the economy. The element $(XA)_k$ is the quantity of the kth good required to operate the economy at the intensity described by the vector X.

The row vector XB, in an analogous manner, is the output of the economy in terms of the n goods. The element $(XB)_k$ is the quantity of the kth good produced when the economy is operated at the intensity described by the vector X.

Our main focus is upon the change in the economy over time (i.e., growth) and hence we are interested in the relation between XB in time period $t-1$ and XA in time period t. It is clear that the physical laws of conservation require that $XB_{(t-1)} \geq XA_{(t)}$. The outputs of period $t-1$ must be sufficient in each good to provide the inputs in the following time period. The subscripts are dropped heareafter for simplicity.

There are standard economic relationships over time that will be useful. It is assumed that there is a rate of interest b from which we can form an interest factor $\beta=(1+b)$ and that there is a rate of growth a from which we may form a growth factor $\alpha=(1+a)$.

With these definitions we may state our objective which is to find vectors X and Y and scalars α and β that will satisfy the following five conditions:

(1) $$XB \geq \alpha XA \qquad \text{or} X(B-\alpha A) \geq 0.$$

If the rate of growth is to be sustained, the outputs must be at least as great as the inputs that will be required in the following time period. Here 0 is the 1 by n row vector containing all zero elements. Every element of the row vector $X(B-\alpha A)$ must be greater than or equal to zero.

(2) $$BY \leq \beta AY \qquad \text{or} (B-\beta A)Y \leq 0.$$

AY and BY are, respectively, the gross values of the inputs and outputs of the m processes. The column vector $(b - \beta A)Y$ must be less than or equal to the column vector 0 consisting of all zeros. The second condition implies that the economy behaves as a perfectly competitive one and prices are adjusted so that no process yields a profit greater than the rate of interest.

(3) $$X(B - \alpha A)Y = 0.$$

This condition requires that the price of an over-produced good be zero. The scalar $X(B - \alpha A)Y$ must equal the scalar zero.

(4) $$X(B - \beta A)Y = 0.$$

The fourth condition requires that processes that do not yield a profit equal to the rate of interest must be used at a zero level of intensity. Here 0 is a scalar.

(5) $$XBY > 0.$$

Every allowable combination of intensity vectors and price vectors must yield outputs that have positive value. XBY is a scalar and must be greater than the scalar zero.

Kemeny, Morgenstern and Thompson [5] have shown that with these five conditions α must be equal to β and that conditions 3 and 4 are redundant. Solutions that satisfy conditions 1, 2, and 5 also satisfy 3 and 4. They have also shown that if every row of A contains at least one positive element and every column of B contains at least one positive element then there exists at least one solution α, X, Y. They further show that there may be more than one such solution and specifically that at most the number of allowable α's for given A and B will be the $\min[m, n]$. (See also [4].)

The generalized von Neumann model is easily adapted to the handling of pollutants as the normal byproducts of economic activity. The m processes may include both "clean" and "dirty" methods. A nominal adjustment will permit us to include p goods and q pollutants within the n products. For some purposes we can add r "eliminated pollutants" so that the number of columns in the A and B matrices will be $n = p + q + r$. Thus we can use all the theorems of the generalized von Neumann model without change.

SOLUTIONS TO SPECIFIC VON NEUMANN SYSTEMS

The existence proofs of [5] are based on the fact that conditions 1, 2, and 5 above define a game with the vector X describing the strategy of the row player and the vector Y the strategy of the column player. As is well known, any rectangular game may be transformed into a linear program. A linear program representing the game described by conditions (1), (2), and (5) is as follows:

Maximize v,

Subject to:

(6) $$\sum_{k=1}^{m} x_k(b_{kj} - \alpha a_{kj}) - v \geq 0, \qquad (j = 1, \ldots, n),$$

$$\sum_{k=1}^{m} x_k = 1. \tag{7}$$

When the optimum value of the objective function (i.e., v) is equal to zero a solution to the von Neumann model has been found; α is an expansion factor of the economy represented by the matrices A and B and the values of x_k ($k=1, \ldots, m$) are the normalized values of the intensity vector X. In this solution, the simplex multipliers corresponding to the first n rows of the linear program are the normalized values of the price vector Y. Thus solving a von Neumann system is the equivalent of finding a value, or values, of α such that the solution of the linear program of (6) and (7) is $v=0$.

Hamburger, Thompson, and Weil [3] have shown that v is a function of α and is monotonic non-increasing. Thus we may compute the value, or values of α, X, and Y by using a simple search technique and solving a series of linear programs iteratively.

The author has written a computer program using the algorithm of reference [3]. The examples that follow have been solved using this program.

An Example of a von Neumann Growth Model with Pollutant Outputs

As a first illustration of the von Neumann growth model with pollutant outputs we will adapt the data of Leontief's example of a national economy as described in Tables 1 through 6 of [8]. Tables I and II contain the matrices A_1 and B_1, which represent this economy with pollutant outputs.

TABLE I. A_1 *Matrix (Input)*

Processes	Goods			
	Wheat	Cloth	Labor	Pollutants
Agriculture	25	14	80	0
Manufacturing	20	6	180	0
Consumption	55	30	0	0

TABLE II. B_1 *Matrix (output)*

Processes	Goods			
	Wheat	Cloth	Labor	Pollutants
Agriculture	100	0	0	50
Manufacturing	0	50	0	10
Consumption	0	0	260	0

There are three goods (wheat, cloth, and labor) and one pollutant measured in grams released. There are three processes: agriculture, manufacturing, and consumption. The consumption process produces labor. We assume, as did Leontief, that the consumption-labor process produces no pollutants in order to simplify the example.

Solving the problem established by the matrices A_1 and B_1 we obtain,

$$\alpha = 1$$
$$X = (0.333, 0.333, 0.333) \text{ and}$$
$$Y = \text{column } (0.25, 0.625, 0.125, 0).$$

The economy will neither expand nor contract. There will be no profits and no savings or net investment. This is exactly what we expect for the consumption-labor process is precisely the final demand vector of Leontief's example and the goods it produces are precisely those required to maintain the system.

The X and Y vectors require some interpretation. The X vector indicates that each process is to be operated at an equal level of intensity. If we calculate the element $(XA)_3$ we find that the labor required at intensity 0.333 is 86.667 and hence if we wish a full employment economy we must multiply the X vector by the "full employment" scalar of 3 to use up the 260 units available.

The price vector Y is converted into precisely that found by Leontief by setting the price of labor to 1 (i.e., dividing through by 0.125). Thus we find that wheat sells for \$2.00 per bushel and cloth \$5.00 per yard. Pollution is "over-produced" and is assigned a price of zero.

The gross outputs (remembering to multiply the X vector by the full-employment scalar) are $3(XB)$ or 100 bushels of wheat, 50 yards of cloth, 260 units of labor, and 60 grams of pollution. These adjusted results are contained in Table III.

TABLE III. *Adjusted results of example 1*

Process intensities: Agriculture, 1;
Manufacturing, 1;
Consumption, 1

Product	Price	Input	Output
Wheat	\$2.00	100	100
Cloth	5.00	50	50
Labor	1.00	260	260
Pollutant	0	0	60

In this example economy there can be no lasting reduction in pollution without a change in the available process technology. Any diversion of resources to pollution control without such a change will induce decline. It is conceivable, but unlikely, that the polluting processes can be cleaned up so they produce less pollutants while continuing to produce the same goods as output in return for the same inputs. The consumption-labor process might be altered. That is, the populace might be induced to produce more labor units for the same units of wheat and cloth in return for the reduction in the level of pollution. Perhaps they can be convinced that their income is to be increased by the value of the good, "eliminated pollutant."

Professor Leontief added a pollution elimination process to his example that requires two units of labor and 0.2 yard of cloth to eliminate 1 gram of pollutant. With his modified input-output model he then calculated that the consumption-labor sector would have to produce 361.8 units of labor (for the same quantities of wheat and cloth) in order to reduce the net pollutants released to 30 grams per cycle [8]. This result can be confirmed with the use of the von Neumann model by making the appropriate changes in the process rows, but there seems no direct way to have calculated these values by using the von Neumann model.

An Example of a von Neumann Model with Pollution Controls

In this example we will examine the effect of government imposed pollution controls. We will con-

tinue to use the input coefficients of Leontief's example, but to avoid the difficulties and the gloomy appearance of an economy with zero growth, we will arbitrarily increase the output rows of the Agriculture and Manufacturing processes by 10 percent. That is, we will assume technological innovations that result in 10 percent more goods and 10 percent more pollutants from these processes. Making these adjustments to the B_1 matrix, we find that the economy is now capable of a 6.2 percent annual growth rate. The adjusted results are contained in Table IV.

TABLE IV. *Adjusted results for increased output—No controls*

Process intensities: Agriculture, 0.998;
Manufacturing, 100.08;
Consumption, 106.196

Product	Price	Input	Output
Wheat	$1.86	103.37	109.78
Cloth	4.74	51.84	55.05
Labor	1.00	260.0	276.11
Pollutant	0	0	65.9

We now add a good and a process. The good will be the removal of pollution (like the removal of garbage). We will assume that the pollution is truly removed although in the real world garbage is not truly eliminated, but simply concentrated in another location or converted to another form of pollution. In the same vein, the pollution removal process will create no pollution directly, although its demand for input goods will bring about additional pollution. In order to insure that the pollution removal industry is used, we will assume that the government requires that the Agriculture and Manufacturing processes reduce the net output of pollution through the expedient of requiring them to purchase one unit of eliminated pollutant for every five units of pollutant they produce.

The matrices A_2 and B_2 in Tables V and VI represent the new technological possibilities. The elements of the pollution removal rows in A_2 and B_2 were derived from Leontief's example [8]. The two polluting industries continue to use the same processes (with outputs increased 10 percent as before), but they are required to buy the appropriate quantities of eliminated pollutant.

We find that the matrices A_2 and B_2 yield $\alpha = 1.00752$ and the economy is capable of growing at the rate of 0.75 percent per cycle. The results, adjusted again for employment at the level of 260 units of labor, are summarized in Table VII. It should be emphasized that the reduction in growth rate from 6.2 to 0.75 percent is a function of the cost of the pollution elimination process.

Net pollutant generated is output pollutant less output eliminated pollutant or 46.65 units. This is 71 percent of the pollutants generated in the economy prior to the introduction of controls. Thus the controls cut pollution by more than the 20-percent rate in the two polluting industries. This is due in part to the reduction in the growth rate from 6.2 to 0.75 percent. The price of wheat rises relative to the price of cloth because Agriculture must buy 0.10 units of eliminated pollutatn while manufacturing need buy only 0.04 unit per yard of cloth.

TABLE V. A_2 *Matrix (input)*

Processes	Goods				
	Wheat	Cloth	Labor	Eliminated pollutant	Pollutant
Agriculture	25	14	80	11.0	0
Manufacturing	20	6	180	2.2	0
Consumption	55	30	0	0	0
Pollution removal	0	1	10	0	5

TABLE VI. B_2 *Matrix (output)*

Process	Goods				
	Wheat	Cloth	Labor	Eliminated pollutant	Pollutant
Agriculture	110	0	0	0	55
Manufacturing	0	55	0	0	11
Consumption	0	0	260	0	0
Pollution removal	0	0	0	5	0

TABLE VII. *Adjusted results for increased output—pollution control at rate of 20 percent*

Process intensities: Agriculture, 0.878;
Manufacturing, 0.923
Consumption, 1.00752;
Pollution Removal, 2.355

Product	Prices	Input	Output
Wheat	2.12	95.83	96.55
Cloth	4.71	50.41	50.79
Labor	1.00	260.00	261.96
Eliminated pollutant	2.97	11.68	11.78
Pollutant	0	11.77	58.43

GROWTH AND POLLUTION CONTROL

The examples of the preceding section illustrate that the goals of economic growth and of pollution control compete for the same scarce resources. In effect, pollution control adds the good, eliminated pollutant to the consumption bill of fare and, in the absence of compensating technological change, savings and investment are reduced accordingly. As savings and investment are reduced, the rate of growth must fall.

To emphasize the point, a series of systems was solved in which the rate of pollution elimination was varied. In the second example of the preceding section, the two polluting processes were required to purchase the good, eliminated pollutant, at the rate of 20 percent of the gross pollutant released. The matrices of Tables V and VI were used in all the systems of this section except that the percentage rate of reduction was varied. That is, matrix A_2 was altered for each case so that the amount of eliminated

pollutant required as input (rows 1 and 2, Table V) was varied as a percent of output pollutant (rows 1 and 2, Table VI). Cases for 0 and 20 percent were solved in the previous section. Additional cases for 10, 40, 60, and 90 percent were solved for this section. Table VIII summarizes the results.

TABLE VIII. *Variations in the rate of pollution elimination*

Percent elimination	α	Net released pollutant
0	1.06196	65.9
10	1.03282	55.44
20	1.00752	46.65
40	0.96514	32.56
60	0.93033	21.58
90	0.88802	8.85

The intensity vector X and the price vector Y may be considered as an economic program that works through the available technology to produce growth at the rate $\alpha - 1$. The quantity of net released pollutant was calculated in each case for "Full employment" at 260 units of labor. That is, it was assumed that the necessary units of wheat and cloth could be obtained to initiate the program at a level employing all 260 units of labor. Figure 1 plots the growth rates and the units of net released pollutant of Table

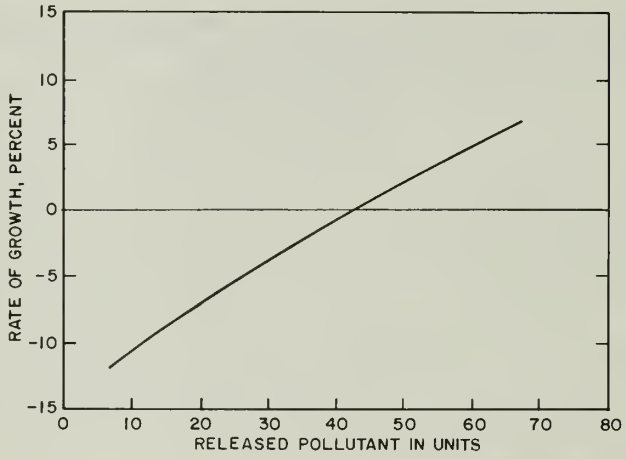

FIGURE 1. Rate of growth vs net release of pollutant in units—Year 1.

VIII. The curve of Figure 1 is an isoquant reflecting the choices between growth and net released pollutants, given the available technology and full employment of available labor resources.

The growth-pollutant trade-off curve of Figure 1 is time dependent. The rate of growth for the ith economic program remains constant, but the absolute quantity of net released pollutant increases at that rate just as do all other outputs. Hence the net released pollutants for time period t will be

(8) $$N_t = (\alpha_i)^{t-1}(P_1 - E_1) \qquad \text{or } N_t = \alpha_i^t N_0.$$

where

N_t = net released pollutant in time period t;

α_i = the growth factor for the ith economic program;

P_1 = gross released pollutant in the first time period (in this case $[XB]_4$);

E_1 = eliminated pollutant in the first time period (in this case $[XB]_5$).

Thus an economic program implies a growth rate and a stream of future goods and bads.

Figure 2 shows a family of growth-pollution trade-off curves for time periods 1, 3, and 5. These

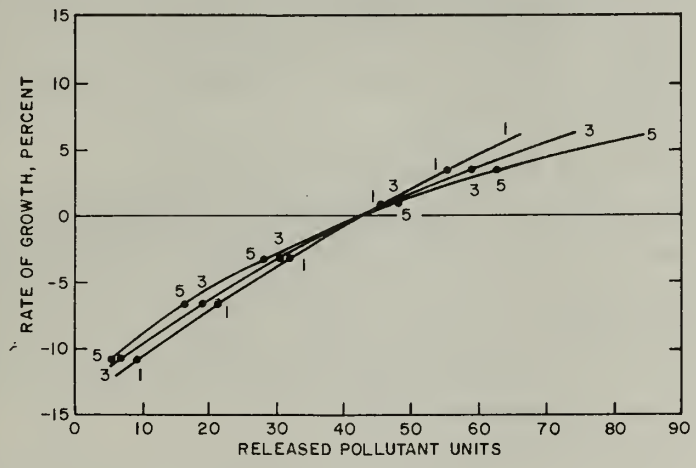

FIGURE 2. Rate of growth vs net release of pollutant in units—Years 1, 3, and 5.

curves pivot on the zero growth path as those programs with positive growth rates result in increasing levels of pollution while those with negative growth rates move in the opposite direction. In selecting an economic program, the present generation chooses a growth-pollution path that affects future generations. The problem is clearly similar to that of trading-off present consumption against future consumption.

If released pollutant were a good rather than a bad, we might calculate the present value of the future stream of goods. Analogously, we might define a function $\phi(N_t)$ that expresses the cost (or disutility) of net released pollutant in any time period t. The function $\phi(N_t)$ may be an engineering estimate of the cost of pollution or it may be a utility function expressing the disutility attached to the level of pollution indicated by N_t. Using the conventional approach, the present cost of the future stream of pollutants is then

9) $$PC = \sum_{t=1}^{T} \frac{\phi(N_t)}{(1+r)^t} = \sum_{t=1}^{T} \frac{\phi\{\alpha^t N_0\}}{(1+r)^t},$$

where

PC = the present cost of the future stream of pollutants;

N_t = the net released pollutant;

r = the appropriate discount rate;

T = the planning horizon.

If we use this approach, we must then choose the proper value for r, the discount rate. If we choose b, as determined by the von Neumann model, then $(1+b)=\beta=\alpha$ and for linear forms of the function $\phi(N_t)$ the present cost (or disutility) of the future stream of pollutants is independent of the rate of growth.

Ramsey [10], in his pioneering study of growth, objected, on ethical grounds, to discounting future consumption. There are similar strong arguments against discounting the future stream of pollutants. In retrospect, perhaps Ramsey's future consumption should have been discounted to compensate for the enormous improvements in available technology since Ramsey's writings were dated 1928.

The von Neumann model is, of course, based on a fixed technology. Many of us believe that research will find less expensive ways to reduce pollution. Thus the proper value of r may be the anticipated rate of reduction in the cost of pollution elimination. In any event, it would seem that the present cost (or disutility) of the future stream of pollutants is a fruitful area for study and that existing economic concepts may be adapted to that study.

EXOGENOUS DEMAND FOR ELIMINATED POLLUTANT

An exogenous demand for eliminated pollutant may be introduced in the manner described in [5] for the introduction of exogenous consumer demand. The following is a brief summary of the method.

Exogenous demand is described by the row vector $d=(d_1, \ldots, d_n)$, where d_k is the exogenous demand for the kth good (or bad). The vector D is always a constant fraction of the output of the system and it expands at the rate $\alpha-1$. Exogenous demand is satisfied in period t by the output of period $t-1$, therefore conditions (1), (2), and (5) become

$$(1^*) \qquad X(B-\alpha A) \geqslant \alpha D$$

$$(2^*) \qquad (B-\beta A)Y \leqslant \beta E(DY)$$

$$(5) \qquad XBY > 0.$$

Conditions (3) and (4) are altered analogously. The vector E in condition (2) is a column vector of dimension m with all elements equal to 1.

Conditions (1*) and (2*) can be expressed as,

$$(1^*) \qquad X[B-\alpha(A+ED)] \geqslant 0,$$

$$(2^*) \qquad [B-\beta(A+ED)]\,Y \leqslant 0.$$

Here E is a column vector of dimension m so that ED is an m by n matrix with all rows consisting of the vector D. Note that $XED=(\Sigma x_i)\,D=D$. We may therefore set $A^*=A+ED$ and, by substituting the matrix $(B-\alpha A^*)$ for our previous matrix $(B-\alpha A)$, solve the system with exogenous demand using the same procedures.

Observe that D represents relative exogenous demand. That is, $XED=D$ in the system before the intensities are scaled to their full employment levels. Letting $\bar{D}$ equal the absolute amount of exogenous demand we see that $D=\bar{D}/\Phi$, where Φ is the full employment scalar.

We now adjust the A_2 matrix of Table V removing the process-oriented purchase of eliminated pollutant and incorporating an exogenous demand for eliminated pollutant of the vector $D = (0, 0, 0, 2.28, 0)$. The revised matrix is shown in Table IX. Matrix B_2 is unchanged as in Table VI.

TABLE IX. A_2^* *Matrix revised for exogenous demand*

Processes	Goods				
	Wheat	Cloth	Labor	Eliminated pollutant	Pollutant
Agriculture	25	14	80	2.28	0
Manufacturing	20	6	180	2.28	0
Consumption	55	30	0	2.28	0
Pollution removal	0	1	10	2.28	5

We find that the revised matrix A_2^* and the original matrix B_2 yield $\alpha = 1.0069$ and the results, adjusted to full employment at 260 units are shown in Table X.

TABLE X. *Adjusted results for exogenous demand example*

Process intensities: Agriculture, 0.876;
Manufacturing, 0.923;
Consumption, 1.0069;
Pollution Removal, 2.382.

Product	Price	Input	Output
Wheat	1.88	95.74	96.40
Cloth	4.73	50.39	50.74
Labor	1.00	260.00	261.80
Eliminated pollutant	5.49	11.83	11.91
Pollutant	0	11.91	58.35

The intensities, inputs and outputs are approximately the same as those shown in Table VI. A more precise scaling of the exogenous demand vector D would reduce these small differences. The prices of the goods are, however, quite different. This is because the effect of the exogenous demand vector is to charge each process for 2.28 units of eliminated pollutant when the process is operated at the unit level. For example, Pollution Removal pays for 0.456 unit of eliminated pollutant per unit of output whereas Agriculture pays for only 0.021 unit per unit of its output. The prices are thus a function of the scale of the process. That is, the incidence of the cost of exogenous demand for eliminated pollutant can be adjusted by scaling the rows of the original A and B matrices. One procedure might be to divide each row of the matrices by the sum of the units of labor in its input and output.

An Example of a von Neumann Growth Model with Alternative Processes

To illustrate the flexibility of the model we will now introduce several changes. We will allow for more than one type of pollutant and process to remove them. We will introduce an alternative manufacturing process with lower pollutant outputs. We will attempt to internalize the effects of pollution

by varying labor units produced. Tables XI and XII contain the A_3 and B_3 matrices for this example.

Pollutants, formerly lumped together, have been separated into air pollutants and water pollutants with corresponding eliminated pollutants and pollutant removal processes. An innovation in the manufacturing process has permitted an alternate process that uses an additional 4 yards of cloth, but reduces emitted pollutants to five units when the process is operated at the unit level. In matrices A_3 and B_3, Manufacturing (a) is the original (dirty) process, while Manufacturing (b) is the alternative (cleaner) process.

Rows 4, 5, and 6 are points on a five dimensional indifference surface: Wheat, Cloth, Labor (leisure foregone), Eliminated Pollutant (air), and Eliminated Pollutant (water). With the consumption of wheat and cloth held constant, workers are willing to produce more labor in return for more eliminated pollutant. Thus workers trade leisure for eliminated pollutant and remain on the same indifference surface. There is no government order to reduce pollution in this example, but the model will choose the trade-off maximizing growth.

We find that matrices A_3 and B_3 yield $\alpha = 1.02733$. The economy is capable of growing at the rate

TABLE XI. A_3 *Matrix (Input)*

Processes	Goods						
	Wheat	Cloth	Labor	Eliminated pollutant (air)	Eliminated pollutant (water)	Air pollutant	Water pollutant
Agriculture	25	14	80	0	0	0	0
Manufacturing (a)	20	6	180	0	0	0	0
Manufacturing (b)	20	10	180	0	0	0	0
Consumption (a)	55	30	0	8	12	0	0
Consumption (b)	55	30	0	5	2	0	0
Consumption (c)	55	30	0	0	2	0	0
Pollutant removal (air)	1	3	4	0	0	5	0
Pollutant removal (water)	0	1	10	0	0	0	5

TABLE XII. B_3 *Matrix (output)*

Processes	Goods						
	Wheat	Cloth	Labor	Eliminated pollutant (air)	Eliminated pollutant (water)	Air pollutant	Water pollutant
Agriculture	110	0	0	0	0	0	55
Manufacturing (a)	0	55	0	0	0	11	0
Manufacturing (b)	0	55	0	6	0	11	0
Consumption (a)	0	0	290	0	0	0	0
Consumption (b)	0	0	260	0	0	0	0
Consumption (c)	0	0	230	0	0	0	0
Pollutant Removal (air)	0	0	0	5	0	0	0
Pollutant Removal (water)	0	0	0	0	5	0	0

2.73 percent per cycle. Consumption (b) was chosen and the results, again adjusted for the employment of 260 units of labor, are summarized in Table XIII.

Both the clean and the dirty manufacturing processes are used. The clean process is less profitable than its alternative for the production of cloth, but more profitable than its alternative for the production of eliminated air pollutant. Thus the clean process is used to the point that it satisfied the demand for 5.27 units of eliminated air pollutant.

TABLE XIII. *Adjusted results for alternative processes example*

Process intensities: Agriculture, 0.934;
Manufacturing (a) ("dirty"), 0.127;
Manufacturing (b) ("clean"), 0.879;
Consumption (a), 0;
Consumption (b), 1.027;
Consumption (c), 0;
Pollutant Removal (air), 0;
Pollutant Removal (water), 0.422.

Product	Price	Input	Output
Wheat	1.74	99.96	102.69
Cloth	4.52	53.86	55.33
Labor	1.00	260.00	267.10
Eliminated air pollutant	3.10	5.14	5.27
Eliminated water pollutant	2.98	2.05	2.11
Air pollutant	0	0	11.07
Water pollutant	0	2.11	51.35

We might derive a family of growth-pollution isoquants for the technology represented by the matrices A_3 and B_3 but we will need new techniques or assumptions. Since we have more than one pollutant, we no longer have the two dimensional plane of Figures 1 and 2. We can perform a partial analysis for each of the pollutants separately or we can define a utility function that reduces the vector of pollutant amounts to a scalar value. In this somewhat more realistic example, we have a multi-dimensional problem in time and in the type of pollutants encountered.

The example illustrates the limitations of the national good eliminated pollutant as an analytical device. Suppose that the cleaner manufacturing process (row 3) uses a precipitator in the smoke stacks of the textile plant, while the air pollutant removal process (row 7) is the service of air filtration in the homes, cars and workplaces of the populace (i.e., very like air "conditioning"). The effect of the air filtration method is to create purified air within a restricted space. That is, it creates clean air as a private good. The smoke stack precipitator removes the air pollutant at its source and hence makes the air of the whole community cleaner. That is, it creates cleaner air as public good. The precipitator can eliminate the need for air filtration service only if it eliminates enough pollutant to hold net released pollutants to a tolerable level.

LIMITING THE TOTAL OUTPUT OF POLLUTANTS

In the final example, the constrained game technique described in [3] is used to show the effect of a constraint on the total net output of air pollution. Charnes has shown that a game in its linear programming form may be constrained by linear inequalities in addition to those implicit in the rela-

tionships between the two players [2]. That is, we may expand the five conditions of the generalized model by adding constraints of the form:

$$XB \leqq F \tag{10}$$

$$XA \leqq G \tag{11}$$

Where X, A, and B are as previously defined, and F and G are row vectors of dimension n.

The linear program to be solved iteratively as described above is of the form:

Maximize v

Subject to:

$$\sum_{k=1}^{m} x_k(b_{kj}-\alpha a_{kj}) - v \geqslant 0 \qquad (j=1, \ldots, n) \tag{6}$$

$$\sum_{k=1}^{m} x_k = 1 \tag{7}$$

$$\sum_{k=1}^{m} x_k b_{kj} \leqq f_j \qquad (j=1, \ldots, n) \tag{12}$$

$$\sum_{k=1}^{m} x_k a_{kj} \leqq g_j \qquad (j=1, \ldots, n). \tag{13}$$

The relations (12) limit the output of the various products and the relations (13) limit the inputs. Of course, not every input or output need be limited. In this example we will constrain only the output of air pollutants. The appropriate values for the elements f_j and g_j are not easily determined because the vector X is a probability vector whose elements sum to 1 and hence indicates only the relative intensities at which the processes are operated. Thus the linear program shown immediately above must be modified to express the pollutant output limit as a ratio.

In the previous examples, we have scaled the vector X so as to reflect a "full employment" level of activity. That is, we have adjusted the results by the full employment scalar Φ so as to set labor inputs equal to 260, the amount of labor assumed to be available. Thus, in the case of the outputs of pollutant and the input of labor, relations (12) and (13) must be scaled by Φ in order that the right-hand sides be expressable in terms of the effect upon the example economy as

$$\sum_{k=1}^{m} \Phi\, x_k b_{kp} \leqq f_p, \tag{14}$$

$$\sum_{k=1}^{m} \Phi\, x_k a_{kh} = g_h, \tag{15}$$

where p and h are the indices of air pollutant and labor, respectively.

The full employment scalar Φ is an unknown and we must eliminate it by dividing the inequality (14) by the equality (15). Cancelling Φ and rearranging terms, we have

$$\sum_{k=1}^{m} x_k\left(b_{kp} - \frac{f_p}{g_h} a_{kh}\right) \leqq 0. \tag{16}$$

Constraint (16) requires that the output of air pollutant be no greater than f_p/g_h times the input of labor, a ratio that we can calculate easily from our basic data. We can, of course, generate additional constraints on the outputs or the inputs of the system using a similar technique. There is no guarantee, however, that a positive growth rate will result.

If our system includes a process producing eliminated pollutant as in the last example, we may constrain the net output of a particular pollutant by the use of a constraint like

$$\sum_{k=1}^{m} x_k\left(b_{kp} - b_{ke} - \frac{f_p}{g_h} a_{kh}\right), \leqq 0, \tag{17}$$

where p and h are the indices of the pollutant and labor as before and e is the index of the product eliminated pollutant.

Tables XIV and XV contain the matrices for a simple example of this technique. A single additional constraint in the form of inequation (16) was set up so as to require that the output of air pollution be less than or equal to 6 units when labor input was equal to 260. The results are shown in Table XVI. The growth rate in this example was 3.711 percent.

TABLE XIV. A_4 *Matrix (input)*

Processes	Goods				
	Wheat	Cloth	Labor	Air pollutant	Water pollutant
Agriculture	25	14	80	0	0
Manufacturing(a)	20	6	180	0	0
Manufacturing(b)	20	10	180	0	0
Consumption	55	30	0	0	0

TABLE XV. B_4 *Matrix (output)*

Processes	Goods				
	Wheat	Cloth	Labor	Air pollutant	Water pollutant
Agriculture	110	0	0	0	55
Manufacturing(a)	0	55	0	11	0
Manufacturing(b)	0	55	0	6	0
Consumption	0	0	260	0	0

TABLE XVI. *Adjusted results with pollutant output constrained*

Process intensities: Agriculture, 0.955;
Manufacturing(a), 0.150
Manufacturing(b), 0.870;
Consumption, 1.037

Product	Prices	Input	Output
Wheat	1.787	101.321	105.081
Cloth	5.080	54.086	56.093
Labor (as input)	0.922	260.000	–
Labor (as output)	1.000	–	269.649
Air Pollutant (as input)	0.000	0.000	–
Air Pollutant (as output)	−3.512	–	6.000
Water pollutant	0.000	0.000	52.541

In our previous examples the prices were the simplex multipliers corresponding to each of the products, but with the addition of a constraint on the total output of air pollutant the calculation of process becomes more complex. The simplex criterion for the ith process vector in the basis of the optimum solution to the model is

$$V_i = 0 = \sum_{\substack{k=1 \\ k \neq h,p}}^{n} \pi_k(b_{ik} - \alpha a_{ik}) + \pi_h(b_{ih} - \alpha a_{ih}) + \pi_p(b_{ip} - \alpha a_{ip}) - \pi_{n+1} - \pi_{n+2}\left(b_{ip} - \frac{f_p}{g_h} a_{ih}\right) - c_i,$$

where p is the index of air pollutant, h is index of labor, π_{n+1} is the simplex multiplier of the constraint requiring the sum of the x_k's to be equal to one, and π_{n+2} is the simplex multiplier of the constraint on total output of air pollutant. Recalling that π_{n+1} will be equal to the value of the game and hence zero in the optimum and that $c_i = 0$ for a process we may rearrange the terms as

$$V_i = 0 = \sum_{\substack{k=1 \\ k \neq h,p}}^{n} \pi_k b_{ik} - \sum_{\substack{k=1 \\ k \neq h,p}}^{n} \alpha \pi_k a_{ik} + \left(\pi_{n+2} \frac{f_p}{g_h} - \alpha \pi_h\right) a_{ih} + \pi_h b_{ih} - \alpha \pi_p a_{ip} + (\pi_p - \pi_{n+2}) b_{ip}.$$

Thus, with the exception of labor as input and air pollutant as output, prices are calculated as before, i.e., input prices are derived by dividing the combined terms by $-\alpha$. As is to be expected, the price of air pollutant as output is π_p (normally equal zero) less the simplex multiplier on its total output constraint. That is, π_{n+2} is an effluent charge on the production of air pollutant.

The price of labor input is perhaps not so easily explained. Labor input has its expected multiplier, $-\alpha \pi_h$ and in addition receives a subsidy of $\pi_{n+2} f_p/g_h$. Mathematically it is clear that output is constrained so that pollution can be no greater than f_p/g_h times input labor. This artificiality is caused by the fact that total air pollution is not in reality limited by the amount of input labor, but rather by the capacity of the atmosphere. Were we able to include this atmospheric process within our model we would indeed wish to subsidize its use. Since we use input labor as a surrogate, the subsidy is placed upon it. Thus the subsidy is not as surprising as it seems at first light. The revenue produced by the

effluent charges must be reflected in the economic program on the input side and in this case it appears as reduction in the cost of labor to other processes.

The case in which there is more than one pollutant to be constrained may be worked out in an analogous manner. Any of the input products may be chosen as limiting the scale of the economy. Other inputs may be constrained in proportion to the limiting input.

CONCLUSIONS

The von Neumann model of an expanding economy has many well known limitations. The assumption of constant returns to scale is particularly troublesome for the study of pollution since both its creation and its abatement have distinctly nonlinear properties. Most pollution problems are local in character and may not be represented easily in the model. Appropriate data for the von Neumann model seem difficult to obtain. Indeed there are many problems that must be solved if the model is to be applied to specific pollution problems.

The von Neumann model does have great flexibility and power, however. The examples have shown that clean and dirty processes, goods, and pollutants are easily handled. The effect of pollution abatement on the price structure is an immediate result of the solution. Direct calculation of effluent charges would seem to be very useful. Perhaps these advantages will make it profitable to overcome the difficulties.

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COVARIANCE OF WEIBULL ORDER STATISTICS

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ABSTRACT

Let $X_1 < X_2 < \ldots < X_n$ denote an ordered sample of size n from a Weibull population with *cdf* $F(x) = 1 - \exp(-x^\rho)$, $x > 0$. Formulae for computing $\mathrm{Cov}(X_i, X_j)$ are well known, but they are difficult to use in practice. A simple approximation to $\mathrm{Cov}(X_i, X_j)$ is presented here, and its accuracy is discussed.

INTRODUCTION

The Weibull distribution is considered a suitable model in many physical situations. A large class of problems of inference relating to the Weibull distribution requires the knowledge of variances and covariances of order statistics. For example, Basu and Lochner [1] used variances and covariances of order statistics from the Weibull distribution to study the robustness of the exponential life testing procedure when the real populations were Weibull.

In this note, formulas are given in section 1 for the exact values of the variances and covariances of the order statistics from the Weibull distribution. To simplify computation a simple approximation for the covariances was obtained and its accuracy was investigated in section 2.

1. EXACT VALUE

Let $X_1 < X_2 < \ldots < X_n$ denote an ordered sample of size n from a Weibull population with *cdf*

$$F(x) = 0, \quad x \leq 0,$$

$$= 1 - e^{-x^\rho} \quad x > 0$$

and hence with *pdf*

$$f(x) = 0, \quad x \leq 0$$

$$= \rho x^{\rho - 1} e^{-x^\rho}, \quad x > 0.$$

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Then Lieblein [3] has shown that

$$(1)\qquad E(X_i^2)=\frac{n!}{(i-1)!(n-i)!}\Gamma(1+2/\rho)\sum_{h=0}^{i-1}(-1)^h\binom{i-1}{h}\cdot(n+h-i+1)^{-1-2/\rho},\qquad 1\leq i\leq n,$$

and

(2)

$$E(X_iX_j)=\rho^2K\sum_{g=0}^{i-1}\sum_{h=0}^{j-i-1}(-1)^{g+h}\binom{i-1}{g}\binom{j-i-1}{h}\cdot\psi(j-i+g-h,\ n-j+h+1),\qquad 1\leq i<j\leq n,$$

where

$$K=n!/((i-1)!(j-i-1)!(n-j)!),$$
$$\psi(t,u)=\rho^{-2}(tu)^{-r}\Gamma(2r)\,B_q(r,r),$$
$$r=1+1/\rho$$
$$q=t/(t+u),$$
$$B_q(r,r)=\int_0^q(y(1-y))^{r-1}\,dy.$$

Equations (1) and (2) were used by the authors to compute tables of $\mathrm{Cov}(X_i, X_j)$, accurate to 10^{-4}, for $\rho=1.5(0.5)3.5$; $n=1,\ \ldots,\ 15$; and $1\leq i\leq j\leq n$.

The computation of $E(X_i^2)$ was carried out in a straightforward manner by using Equation (1) and double precision arithmetic (except for $\Gamma(\cdot)$, which was computed accurate to 5×10^{-10} by using a standard library program). The computation of $E(X_iX_j)$ was considerably more involved. In order to have $E(X_iX_j)$ accurate to within 10^{-4} it was found necessary to compute $B_q(r, r)$ to an accuracy of up to 10^{-14}. Ordinary double precision computations consumed excessive amounts of time, so the procedure used in the computation is as follows. First a table of $B_q(r, r)$ was computed for $\rho=1.5(0.5)3.5$ (where $r=1+1/\rho$) and $q=0.05(0.01)0.95$, accurate to within 10^{-14}. A standard numerical integration routine was used. Then the computation of $E(X_iX_j)$ was carried out using Equation (2) and double precision arithmetic, except for the computation of $B_q(r, r)$. For each (n, i, j) it was determined how accurate each term in the series of Equation (2) had to be if the final computed $E(X_iX_j)$ was to be accurate to 10^{-4}. From this the accuracy requirement for the value of $B_q(r, r)$ was determined. Simple interpolation in the previously prepared $B_q(r, r)$ table could not provide the accuracy usually needed, so numerical integration over small intervals (of less than 0.005 in length) was carried out.

As an example of the method used, suppose we want to compute $E(X_5X_8)$ when $n=10$ and $\rho=2$. Then Equation (2) sums 15 terms. We want each term accurate to $\pm10^{-4}/15$. For the $g=1$, $h=2$ term, then, we want to compute

$$\rho^2K(-1)^{g+h}\binom{i-1}{g}\binom{j-i-1}{h}\psi(j-i+g-h,\ n-j+h+1)=\frac{-9!}{12\sqrt{10}}B_{2/7}(1.5,\ 1.5)$$

accurate to $\pm10^{-4}/15$. Hence we want to compute $B_{2/7}(1.5, 1.5)$ accurate to $\pm(12\sqrt{10}/9!)10^{-4}/15\doteq\pm7\times10^{-10}$. We note that

$$B_{2/7}(1.5, 1.5) = \int_0^{0.2857} (y(1-y))^{0.5}\, dy$$

$$= B_{0.29}(1.5, 1.5) - \int_{0.2857}^{0.2900} (y(1-y))^{0.5} dy.$$

We have $B_{0.29}(1.5, 1.5)$ available already from our table. We obtain the second integral, accurate to $\pm 7 \times 10^{-10}$, by numerical integration.

The covariance tables are available upon request from the authors [4].

2. APPROXIMATION

An asymptotic form for the covariances of order statistics is given by

$$A(n, i, j) = p_i(1 - p_j)/((n+2)\ f(\lambda_i)f(\lambda_j)), \tag{3}$$

where $p_i = i/(N+1)$, $F(\lambda_i) = p_i$, and $f(x) = F'(x)$ is the density function for the parent population [2]. For a Weibull parent population this can be written as

$$A(n, i, j) = [i/(n+2)\,(n+1-i)]\rho^{-2}[-\ln(1-i/(n+1))]^{1/\rho - 1}[-\ln(1-j/(n+1))]^{1/\rho - 1}. \tag{4}$$

This form was compared with the exact covariance values and was found to provide a poor approximation for small values of n. The approximation was almost always less than the exact value. Errors were largest for small n and ρ. For $n = 10$ and $\rho = 1.5$, for example, errors greater than 0.01 were common. Errors of over 0.003 occured for $n = 8$ and $\rho = 2.5$. When comparing asymptotic and exact covariances for $n = 5$, maximum errors of 0.038, 0.008, and 0.004 were observed for $\rho = 1.5$, 2.5, and 3.5, respectively.

It was observed, however, that the ratio of the exact value over the asymptotic value tended to follow a pattern. The possibilities of this ratio, or its inverse or its logarithm, being a linear function of n, i and j were investigated. In order to keep the approximation simple, more involved models were not considered. The authors found from this study that the covariance for the ith and jth order statistics from a sample of size n could be reasonably approximated by

$$\widehat{\mathrm{Cov}}(X_i, X_j) = A(n, i, j) \cdot (a + bn + ci + dj), \tag{5}$$

where a, b, c, and d are given in Table 1 and where $A(n, i, j)$ is as given in (4). This approximation, which was obtained by fitting the equation

$$\mathrm{Cov}(X_i, X_j)/A(n, i, j) = a + bn + ci + dj$$

TABLE 1. *Coefficients for covariance approximation*

ρ	a	b	c	d
1.5	1.3426	−0.0270	0.0006	0.0212
2.0	1.2761	−0.0214	−0.0016	0.0190
2.5	1.2664	−0.0186	−0.0039	0.0171
3.0	1.2721	−0.0169	−0.0059	0.0155
3.5	1.2830	−0.0157	−0.0076	0.0142

TABLE 2. *Covariances of Weibull order statistics*

Sample size	Cov (X_i, X_j)						
n	i	j	$\rho=1.5$	$\rho=2.0$	$\rho=2.5$	$\rho=3.0$	$\rho=3.5$
1	1	1	0.3757	0.2146	0.1442	0.1053	0.0811
2	1	1	0.1491	0.1073	0.0828	0.0664	0.0545
2	2	1	0.1116	0.0674	0.0462	0.0339	0.0261
2	2	2	0.3791	0.1871	0.1132	0.0764	0.0553
3	1	1	0.0868	0.0715	0.0599	0.0506	0.0433
3	2	1	0.0678	0.0478	0.0360	0.0282	0.0227
3	2	2	0.1648	0.0995	0.0678	0.0496	0.0380
3	3	1	0.0539	0.0335	0.0233	0.0172	0.0133
3	3	2	0.1325	0.0709	0.0447	0.0310	0.0228
3	3	3	0.3671	0.1682	0.0968	0.0629	0.0443
4	1	1	0.0592	0.0537	0.0475	0.0418	0.0367
4	2	1	0.0469	0.0367	0.0294	0.0240	0.0199
4	2	2	0.1010	0.0688	0.0504	0.0387	0.0307
4	3	1	0.0390	0.0276	0.0206	0.0161	0.0129
4	3	2	0.0846	0.0521	0.0357	0.0262	0.0200
4	3	3	0.1659	0.0913	0.0586	0.0410	0.0304
4	4	1	0.0321	0.0204	0.0144	0.0107	0.0083
4	4	2	0.0699	0.0388	0.0250	0.0175	0.0130
4	4	3	0.1382	0.0689	0.0415	0.0278	0.0200
4	4	4	0.3552	0.1554	0.0866	0.0551	0.0382
5	1	1	0.0440	0.0429	0.0398	0.0360	0.0323
5	2	1	0.0351	0.0298	0.0250	0.0211	0.0179
5	2	2	0.0710	0.0528	0.0408	0.0324	0.0264
5	3	1	0.0298	0.0229	0.0181	0.0146	0.0120
5	3	2	0.0604	0.0409	0.0297	0.0226	0.0178
5	3	3	0.1047	0.0643	0.0440	0.0322	0.0247
5	4	1	0.0256	0.0182	0.0136	0.0106	0.0085
5	4	2	0.0521	0.0325	0.0225	0.0165	0.0127
5	4	3	0.0906	0.0515	0.0335	0.0237	0.0176
5	4	4	0.1639	0.0850	0.0524	0.0356	0.0259
5	5	1	0.0215	0.0139	0.0099	0.0074	0.0058
5	5	2	0.0438	0.0250	0.0164	0.0116	0.0086
5	5	3	0.0765	0.0398	0.0246	0.0167	0.0121
5	5	4	0.1394	0.0663	0.0388	0.0255	0.0180
5	5	5	0.3451	0.1460	0.0796	0.0499	0.0342
6	1	1	0.0345	0.0358	0.0343	0.0319	0.0291
6	2	2	0.0539	0.0429	0.0345	0.0282	0.0234
6	3	3	0.0751	0.0501	0.0360	0.0272	0.0212
6	4	4	0.1052	0.0605	0.0396	0.0281	0.0210
6	5	5	0.1612	0.0801	0.0480	0.0321	0.0230
6	6	5	0.1389	0.0639	0.0365	0.0236	0.0165
6	6	6	0.3365	0.1388	0.0744	0.0461	0.0313
7	6	6	0.1583	0.0761	0.0447	0.0294	0.0208
7	7	7	0.3292	0.1331	0.0704	0.0433	0.0291
8	8	8	0.3229	0.1283	0.0672	0.0410	0.0274
9	9	9	0.3174	0.1244	0.0645	0.0391	0.0260
10	10	10	0.3125	0.1209	0.0622	0.0375	0.0249
11	11	11	0.3082	0.1180	0.0603	0.0361	0.0239
12	12	12	0.3042	0.1153	0.0586	0.0350	0.0231
13	13	13	0.3006	0.1130	0.0571	0.0339	0.0224
14	14	14	0.2974	0.1110	0.0558	0.0331	0.0217
15	15	15	0.2944	0.1090	0.0546	0.0323	0.0211

by least squares, considerably improved the approximation. It gave a variance approximation accurate to within 0.0052 for $\rho = 1.5$, $n \geq 7$, and to within 0.0033 for $\rho \geq 2.0$, $n \geq 7$. This does not include the cases where $n = i = j$, however. Here the fit was not as good, and the variances for $n = i = j = 1, \ldots, 15$ are given in Table 2. The approximation for covariances ($i \neq j$) was accurate to within 0.0032 for $\rho = 1.5$, $n \geq 8$, and to within 0.0022 for $\rho \geq 2.0$, $n \geq 6$.

The asymptotic formula (4) did not, in general, provide a good approximation for variances, except for $\rho = 3.5$, $n \geq 12$, when the accuracy was within 0.005. Equation (4) provided a better approximation for covariances. Accuracy was within 0.005 for $\rho \geq 2.5$, $n \geq 10$. For $\rho = 1.5$ and 2.0, however, errors as large as 0.0254 were observed for $n = 15$.

It should be mentioned that the special case of $\rho = 1$ (the negative exponential distribution) was excluded from our results because the covariance for $\rho = 1$ is easily obtained using equations found on page 343 of Sarhan and Greenberg [5]. Also, our approximation method provided a poor fit for $\rho = 1$. Tables of exponential covariances for $n \leq 10$ are given in [5].

3. ACKNOWLEDGMENTS

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LARGE DEVIATIONS OF THE WINSORIZED MEAN OF AN EXPONENTIAL DISTRIBUTION

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ABSTRACT

An asymptotic representation for large deviation probabilities of the Winsorized mean of a sequence of independent, identically distributed exponential random variables is derived. The Winsorized mean, a linear combination of exponential order statistics, is first transformed into a weighted sum of exponential random variables, and then a large deviation theorem for weighted sums can be applied. The representation obtained is then compared with results already known for the mean and the median, the two extreme cases of the Winsorized mean.

1. INTRODUCTION

One statistic which estimates the central location of a distribution in the presence of outliers, or seemingly extraneous observations, is the Winsorized mean. The procedure is discussed in, for example, David [5] and Dixon and Massey [6]. The Winsorized mean of a distribution is defined as follows: if $X_1, \ldots, X_n$ is a set of independent, identically distributed (*i.i.d.*) random variables, with order statistics $X_{(1)}, \ldots, X_{(n)}$, where $X_{(1)} \leq \ldots \leq X_{(n)}$, then for each δ, $0 \leq \delta \leq 1/2$, the Winsorized mean is

$$W_n(\delta) = n^{-1}[\delta n] X_{([\delta n])} + n^{-1} \sum_{k=[\delta n]+1}^{n-[\delta n]} X_{(k)} + n^{-1}[\delta n] X_{(n-[\delta n]+1)},$$

where $[y]$ denotes the greatest integer $\leq y$. It can be seen from the definition that the Winsorized mean counts all $[\delta n]$ observations which are $X_{([\delta n])}$ or lower as if they were equal to $X_{([\delta n])}$, and all $[\delta n]$ observations which exceed or equal $X_{(n-[\delta n]+1)}$ as if they were equal to $X_{(n-[\delta n]+1)}$. All other observations are counted as themselves. This procedure has the effect of reducing the influence that extreme observations have on the mean, without completely disregarding them as the median does.

In certain tests for location parameters, the rejection regions have the form $W_n(\delta) > \beta_n$ for some value β_n. Therefore the large deviation probabilities $P(W_n(\delta) > \beta_n)$ under the hypothesis are the significance levels of the tests based on the statistics $W_n(\delta)$, and the tests ordinarily improve as $n \to \infty$ in the sense that the significance levels tend to 0. The Bahadur "exact slope,"

$$(-1/2) \lim_{n \to \infty} n^{-1} \log P(W_n(\delta) > \beta_n),$$

of the tests based on the Winsorized mean is one way of measuring the rate of convergence to 0 of the significance levels. If we have two tests of the same hypothesis, the ratio of their exact slopes provides a comparison of the rates at which their respective significance levels tend to 0; that ratio is called

the Bahadur asymptotic relative efficiency of one test with respect to the other. In the present article, our goal is to determine an asymptotic representation of the large deviation probability $P(W_n(\delta) > \beta_n)$ in the exponential case.

2. PRELIMINARIES

We consider a sequence $\{V_n : 1 \leq n < \infty\}$ of *i.i.d.* random variables having the exponential distribution $G(x) = P(V_1 \leq x)$, where

$$G(x) = 1 - \exp(-x) \qquad \text{for } x \geq 0$$

(1)

$$= 0 \qquad \text{for } x < 0.$$

If $V_{1n}, \ldots, V_{nn}$ are the increasing order statistics of a sample of size n from the exponential distribution (1), i.e., $V_{1n} \leq \ldots \leq V_{un}$, then Rényi [7] showed that each V_{jn} can be expressed as a particular linear combination of *i.i.d.* exponential random variables with distribution function (*d.f.*) (1). More precisely, there exists a sequence $\{Z_n : 1 \leq n < \infty\}$ of *i.i.d.* exponential random variables such that, for every positive integer n and each j, $1 \leq j \leq n$,

$$V_{ju} = \sum_{k=1}^{j} (n-k+1)^{-1} Z_k.$$

If we consider a weighted sum of the order statistics, $W_n = \sum_{j=1}^{n} c_{jn} V_{jn}$, it follows upon interchanging the order of summation that, for $c > 0$,

$$P\left(W_n > (1+c) \sum_{k=1}^{n} d_{nk} \right) = P\left(\sum_{k=1}^{n} a_{nk} X_k > c \sum_{k=1}^{n} a_{nk} \right), \tag{2}$$

where

$$d_{nk} = (n-k+1)^{-1} \sum_{j=k}^{n} c_{jn};$$

$$a_{nk}^2 = d_{nk}^2 \left(\sum_{k=1}^{n} d_{nk}^2 \right)^{-1};$$

and

$$X_k = Z_k - 1.$$

The random variable X_k has *d.f.*

$$F(x) = 1 - \exp(-(1+x)) \qquad \text{for } x \geq -1$$

(3)

$$= 0 \qquad \text{for } x < -1;$$

density

$$f(x) = \exp(-(1+x)) \qquad \text{for } x \geq -1$$

(4) $$= 0 \qquad \text{for } x < -1;$$

and moment-generating function (*m.g.f.*)

(5) $$\phi(t) = (1-t)^{-1} \exp(-t)$$

which exists for $t < 1$ only. We also define

(6) $$Q(t) = \phi'(t)/\phi(t) = t(1-t)^{-1}.$$

For $0 \leq \delta \leq 1/2$, the Winsorized mean of a sample of size n from the exponential distribution (1) is defined by

$$W_n(\delta) = n^{-1}[\delta n] V_{[\delta n],n} + n^{-1} \sum_{j=[\delta n]+1}^{n-[\delta n]} V_{jn} + n^{-1}[\delta n] V_{(n-[\delta n]+1),n}.$$

(In the arguments which follow, δ is a fixed real number between 0 and 1/2, inclusive.) Therefore $W_n(\delta) = \sum_{j=1}^{n} c_{jn} V_{jn}$, where

$$\begin{aligned} c_{jn} &= 0 && \text{for } 1 \leq j \leq [\delta n]-1,\ n-[\delta n]+2 \leq j \leq n \\ &= n^{-1}[\delta n] && \text{for } j = [\delta n],\ j = n-[\delta n]+1 \\ &= n^{-1} && \text{for } [\delta n]+1 \leq j \leq n-[\delta n]. \end{aligned} \tag{7}$$

Because $d_{nk} = (n-k+1)^{-1} \sum_{j=k}^{n} c_{jn}$, we have

$$\begin{aligned} d_{nk} &= (n-k+1)^{-1} && \text{for } 1 \leq k \leq [\delta n) \\ &= n^{-1} && \text{for } [\delta n]+1 \leq k \leq n-[\delta n]+1 \\ &= 0 && \text{for } n-[\delta n]+2 \leq k \leq n. \end{aligned} \tag{8}$$

If we define $D_n^2 = \sum_{k=1}^{n} d_{nk}^2 = n^{-2}(n-2[\delta n]+1) = \sum_{k=1}^{[\delta n]} (n-k+1)^{-2}$, we can write $a_{nk} = D_n^{-1} d_{nk}$. We set $\sigma_n = \max\{a_{nk} : 1 \leq k \leq n\}$. We now obtain several lemmas with a view toward applying the large deviation theorems in Ref. [3].

LEMMA 1: At least $(1-\delta)n$ of the a_{nk}'s exceed or equal $(1-\delta)\sigma_n$.

PROOF: $\sigma_n = D_n^{-1} \max\{d_{nk} : 1 \leq k \leq n\} = D_n^{-1}(n-[\delta n]+1)^{-1}$. Therefore all nonzero a_{nk}'s exceed or equal $D_n^{-1} n^{-1} = n^{-1}(n-[\delta n]+1)\sigma_n$, and there are $n-[\delta n]+1$ nonzero a_{nk}'s. The lemma follows from the observations that $n-[\delta n]+1 \geq (1-\delta)n$ and $n^{-1}(n-[\delta n]+1) \geq 1-\delta$.

LEMMA 2: $n \sum_{k=1}^{n} a_{nk}^4 \leq 14$ for every δ, $0 \leq \delta \leq 1/2$.

PROOF: From (8), we estimate that $D_n^2 \geq n^{-2}(n-[\delta n]+1) \geq (2n)^{-1}$ because $\delta \leq 1/2$. It follows that

$$n\sum_{k=1}^{n} a_{nk}^4 = nD_n^{-4}\sum_{k=1}^{n} d_{nk}^4 \leq n(2n)^2\sum_{k=1}^{n} d_{nk}^4$$

$$\leq (4n^3)n^{-4}(n-2[\delta n]+1)+(4n^3)\sum_{k=1}^{[\delta n]}(n-k+1)^{-4}$$

$$\leq 4n^{-1}(n-2[\delta n]+1)+4n^3\int_{n-[\delta n]}^{n} x^{-4}dx$$

$$\leq 4+(28/3)<14.$$

LEMMA 3: There exists, for each n, a solution $h=h_n$ to the equation

$$\sum_{k-1}^{n} a_{nk}Q(ha_{nk}) = c\sum_{k=1}^{n} a_{nk}, \tag{9}$$

and the solution satisfies the inequalities

$$b_0 = c(c+1)^{-1} \leq h_n\sigma_n \leq c\{c(1-\delta)+(1-\delta)^3\}^{-1} = B_0. \tag{10}$$

PROOF: Consider the function $Q_n^*(h) = \sum_{k=1}^{n} a_{nk}Q(ha_{nk})$. Then $Q_n^*(0)=0$ from (6),

$$Q_n^*(h) \geq \sigma_n Q(h\sigma_n),$$

and $Q_n^*(h)$ is continuous for $0 \leq h \leq 1$. Since Q takes on all positive values, there exists a value of h for which $Q(h\sigma_n) = \sigma_n^{-1}c\sum_{k=1}^{n} a_{nk}$. Continuity then implies the existence of an h_n such that

$$Q_n^*(h_n) = c\sum_{k=1}^{n} a_{nk}.$$

For the upper bound, Lemma 1 implies that

$$c\sum_{k=1}^{n} a_{nk} = \sum_{k=1}^{n} a_{nk}Q(h_n a_{nk}) \geq (1-\delta)^2 n\sigma_n Q(h_n(1-\delta)\sigma_n),$$

since Q is monotonically increasing. But $n\sigma_n\left(\sum_{k=1}^{n} a_{nk}\right)^{-1} \geq n\sigma_n(n\sigma_n)^{-1}=1$, so that

$$c \geq (1-\delta)^2 Q(h_n(1-\delta)\sigma_n),$$

and then $h_n\sigma_n \leq (1-\delta)^{-1}Q^{-1}(c(1-\delta)^{-2})$. The upper bound follows from the observation that $Q^{-1}(t)=t(1+t)^{-1}$. For the lower bound

$$c\sum_{k=1}^{n} a_{nk}=\sum_{k=1}^{n} h_n a_{nk}^2(1-h_n a_{nk})^{-1} \leq h_n(1-h_n\sigma_n)^{-1}\sum_{k=1}^{n} a_{nk}^2=h_n(1-h_n\sigma_n)^{-1}.$$

But $\sigma_n \sum_{k=1}^{n} a_{nk} \geq \sum_{k=1}^{n} a_{nk}^2=1$, so that $c\sigma_n^{-1} \leq h_n(1-h_n\sigma_n)^{-1}$, and $c-ch_n\sigma_n \leq h_n\sigma_n$, yielding $h_n\sigma_n \geq c(c+1)^{-1}$.

LEMMA 4: $\phi(B_0)<\infty$ if $c<\delta^{-1}(1-\delta)^3$.

PROOF: $\phi(B_0)<\infty$ if $B_0<1$. This occurs if $c<c(1-\delta)+(1-\delta)^3$, i.e. if $c<\delta^{-1}(1-\delta)^3$.

LEMMA 5: If $c<\delta^{-1}(1-\delta)^3$, the functions $f_\lambda(z)=e^{\lambda z}f(z)$ have total variations $V_\lambda \leq 1$ for $0 \leq \lambda \leq B_0$.

PROOF: $f_\lambda(z)=\exp(-1-(1-\lambda)z)$ for $z \geq -1$, and 0 otherwise, by (4). Therefore $V_\lambda=e^{-\lambda}$ for $0 \leq \lambda \leq B_0<1$.

3. THE LARGE DEVIATION THEOREMS

We can now apply Theorem (4.8) of Ref. [3] (p. 1229), a large deviation theorem for weighted sums of the form $\sum_{k=1}^{n} a_{nk}X_k$. Lemmas 2–5 guarantee all the conditions that the large deviation theorem requires. Defining $P_n(\delta)=P\left(W_n(\delta) > (1+c)\sum_{k=1}^{n} d_{nk}\right)$, we obtain:

THEOREM 1: If $c<\delta^{-1}(1-\delta)^3$, then

$$P_n(\delta)=(2\pi)^{-1/2}(\gamma_n h_n)^{-1}\exp\left\{-h_n(1+c)\sum_{k=1}^{n} a_{nk}\right\}\left\{\prod_{k=1}^{n}(1-h_n a_{nk})\right\}^{-1}(1+0(\sigma_n)),$$

where $\gamma_n^2=\sum_{k=1}^{n} a_{nk}^2(1-h_n a_{nk})^{-2}$, and $0(\sigma_n)\to 0$ as $n\to\infty$.

It remains to analyze and estimate the several quantities appearing in the formula, for the purpose of obtaining a useful representation of the large deviation probability. From (8), we have

$$\sum_{k=1}^{n} d_{nk}=n^{-1}(n-2[\delta n]+1)+\sum_{k=1}^{[\delta n]}(n-k+1)^{-1}$$

and then $\sum_{k=1}^{n} a_{nk}=\left(\sum_{k=1}^{n} d_{nk}^2\right)^{-1/2}\sum_{k=1}^{n} d_{nk}$. From (10), Lemma 1, and the definition of σ_n, we find that

$$b_0(1-\delta)^2 n \leq h_n\sum_{k=1}^{n} a_{nk} \leq B_0 n,$$

and from (6), (9), and (10), we obtain that

$$cb_0(1-\delta)^2 n \leq (\gamma_n h_n)^2 \leq cB_0(1-B_0)^{-1}n$$

so that

$$(cB_0)^{-1/2}(1-B_0)^{1/2}n^{-1/2} \leq (\gamma_n h_n)^{-1} \leq (cb_0)^{-1/2}(1-\delta)^{-1}n^{-1/2}.$$

From (10) and Lemma 1, it follows that

$$(1-B_0)^n \leq \prod_{k=1}^{n} (1-h_n a_{nk}) \leq \{(1-b_0(1-\delta))\}^{(1-\delta)n}$$

so that

$$(1-B_0)^{-n} \geq \left\{ \prod_{k=1}^{n} (1-h_n a_{nk}) \right\}^{-1} \geq \{(1-b_0(1-\delta))^{(1-\delta)}\}^{-n}.$$

In view of the above estimates, Theorem 1 really asserts that

$$P_n(\delta) \sim (2\pi n)^{-1/2} A_n B_n^{-n} e^{-nC_n},$$

where A_n, B_n, and C_n are bounded sequences (with known bounds), and the symbol "$\sim$" signifies that the ratio of the two sides tends to 1 as $n \to \infty$.

Approximating by integrals, as in the proof of Lemma 2, we obtain the following estimates:

$$\left| \sum_{k=1}^{n} d_{nk} - (1-2\delta-\log(1-\delta)) \right| \leq 3n^{-1}; \tag{11}$$

$$\left| \sum_{k=1}^{n} d_{nk}^2 - n^{-1}(1-2\delta+\delta(1-\delta)^{-1}) \right| \leq 2n^{-2}. \tag{12}$$

From the fact, noted in the proof of Lemma 1, that $\sigma_n = D_n^{-1}(n-[\delta n]+1)^{-1}$, we can write, using (11) and (12), that

$$\sigma_n \sim n^{-1/2}(1-\delta)^{-1}(1-2\delta+\delta(1-\delta)^{-1})^{-1/2}. \tag{13}$$

Because $a_{nk} = D_n^{-1} d_{nk}$, (11) and (12) also imply that

$$\sum_{k=1}^{n} a_{nk} \sim n^{1/2}(1-2\delta+\delta(1-\delta)^{-1})^{-1/2}(1-2\delta-\log(1-\delta)). \tag{14}$$

Combining (13) and (14), we obtain

$$\sigma_n \left(\sum_{k=1}^{n} a_{nk} \right)^{-1} \sim n^{-1}(1-\delta)^{-1}(1-2\delta-\log(1-\delta))^{-1}. \tag{15}$$

In view of (15), we can apply Corollary (5.1) of Ref. [3] (p. 1231) to arrive at the following theorem, recalling that, in the present situation, $\alpha = \theta = 1-\delta$:

THEOREM 2: If $c < \delta^{-1}(1-\delta)^3$, then

$$\limsup_{n\to\infty} n^{-1} \log P_n(\delta) \leq -(1-\delta)^2(1-2\delta-\log(1-\delta))L((1-\delta)c(c+1)^{-1}),$$

and

$$\liminf_{n\to\infty} n^{-1} \log P_n(\delta) \geq -(1-\delta)^{-2}(1-2\delta-\log(1-\delta))L((1-\delta)c\{c+(1-\delta)^2\}^{-1}),$$

where $L(u)=u(1-u)^{-1}+\log(1-u)$.

4. TWO SPECIAL CASES

The large deviation probabilities for the two extreme cases of the Winsorized mean $W_n(\delta)$, namely the sample mean ($\delta=0$) and the sample median ($\delta=1/2$), have already been worked out for a large class of distributions. Estimates along the lines of Theorem 1 were obtained for the sample mean by Bahadur and Ranga Rao [1] and for the sample median by the present author [2]. The analogue of Theorem 2 was worked out for the sample mean by Chernoff [4] and for the sample median by Sievers [8].

Chernoff's theorem asserts that, for *i.i.d.* random variables $\{Y_k : 1 \leq k < \infty\}$ with $E(Y_1)=0$,

$$\lim_{n\to\infty} n^{-1} \log P\left(n^{-1}\sum_{k=1}^{n} Y_k \geq c\right) = -\log \rho,$$

where $\rho = \min\{e^{-ct}\phi(t) : t>0\}$, where $\phi(t)$ is the *m.g.f.* of Y_1. Taking $\delta=0$ in Theorem 2 above, we see that $W_n(0)=n^{-1}\sum_{k=1}^{n} V_k$, where $\{V_k : 1 \leq k < \infty\}$ are *i.i.d.* exponential random variables with *d.f.* (1). Since $E(V_1)=1$, we set $Y_k=V_k-1$, and then

$$P_n(0)=P(W_n(0)>1+c)=P\left(n^{-1}\sum_{k=1}^{n} Y_k \geq c\right),$$

where Y_1 has *m.g.f.* (5), because each $d_{nk}=n^{-1}$ when $\delta=0$, according to (8). The minimum of $e^{-ct}\phi(t)$ occurs when $t=c(c+1)^{-1}$, so that $\rho=(1+c)e^{-c}$. This would mean that

$$\lim_{n\to\infty} n^{-1} P_n(0) = -c+\log(1+c)$$

by use of Chernoff's theorem. The same result can be obtained by direct substitution of $\delta=0$ into Theorem 2.

Sievers' result on the sample median claims that, if M_n is the median of an *odd* number n of observations from the exponential distribution (1), then for $\theta > \log 2$ (the median of the exponential distribution),

$$\lim_{n\to\infty} n^{-1} \log P(M_n > \theta) = \log 2 - \theta + 1/2 \log(e^{\theta}-1).$$

If we write $\theta=(1+c)\log 2$, Sievers' result becomes

$$\lim_{n\to\infty} n^{-1} \log P(M_n > (1+c)\log 2) = -c\log 2 + 1/2 \log(2^{1+c}-1). \tag{16}$$

If we set $\delta = 1/2$ in section 3, the theorems there concern

$$P_n(1/2) = P\left(W_n(1/2) > (1+c)\sum_{k=1}^{n} d_{nk}\right),$$

where $W_n(1/2) = 1/2V_{1/2n,\ n} + 1/2V_{1/2+1,\ n}$ is the median of an even number of observations from the exponential distribution. When $\delta = 1/2$, we can reduce the assertions of Theorem 2 to

$$\lim_{n\to\infty} \sup\, n^{-1} \log P_n(1/2) \leq -4^{-1}\left(\frac{c}{c+2}\right)\log 2 + 4^{-1}(\log 2)\,\log\left(\frac{1+c}{1+2^{-1}c}\right),$$

(17)

$$\lim_{n\to\infty} \inf\, n^{-1} \log P_n(1/2) \geq -4\left(\frac{c}{c+2^{-1}}\right)\log 2 + 4(\log 2)\,\log\left(\frac{1+4c}{1+2c}\right).$$

In view of (11) with $\delta = 1/2$, we see that $P_n(1/2)$ is somewhat comparable to $P(W_n(1/2) > (1+c)\log 2)$. We see also that (17) is reminiscent of Sievers' result (16), even though M_n is not exactly the same statistic as $W_n(1/2)$.

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ON MAXIMUM INTERNALLY STABLE SETS OF A GRAPH

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ABSTRACT

The paper makes some remarks on the paper of Hakimi and Frank and shows a simplified way of applying the concept of alternating forest. An algorithm for finding a maximum internally stable set of an undirected graph is constructed and some examples are given.

1. INTRODUCTION AND BASIC CONCEPTS

The determination of a maximum internally stable set has many applications to operations research and related areas. As the large number of references in [1] shows, much work has been done to find a way of determining a maximum internally stable set of an undirected graph. This paper will show how one can use the tool of alternating forest (called alternating tree in [1]), defined by Hakimi and Frank in [1], to construct an algorithm for the determination of a maximum internally stable set. First we shall define some basic notations and make a useful observation. We shall mainly follow the definitions and notations of F. Harary in [3].

$G_0 = (V(G_0), E(G_0))$ denotes an undirected, connected finite graph without loops and multiple edges, where $V(G_0) = \{x, y, z, \ldots\}$ is the set of vertices in G and $R(G_0)$ the set of edges. If an edge joins two vertices x and y in G_0, then $(x, y) \epsilon E(G_0)$. Γx is the set of all vertices y of G_0 adjacent to x.

A set S of vertices in G_0 is called an internally stable (IS) set of G_0, if no two vertices in S are adjacent, i.e., $\Gamma S \cap S = \phi$. Let A be a set of vertices in G_0; by A we denote the complement of A in $V(G_0)$. An IS set S of a graph G_0 is a complete IS set, if there exists no vertex $x \epsilon V(G_0) \cap S$ such that $\Gamma x \cap S = \phi$. If an IS set S is not complete, then there exists a vertex $x_0 \epsilon V(G_0) \cap S$ such that $S \cup x_0$ is an IS set. Thus, one can continue enlarging the resulting IS set until it is complete. An IS set S of G_0 is a maximum internally stable (MIS) set, if there exists no IS set S_1 of G_0 such that $|S_1| > |S|$, where the notation $|A|$ indicates the number of elements in the set A. Clearly a MIS set is a complete IS set of G_0.

The definition of the complete IS set directly implies the following useful lemma:

LEMMA 1: Let G_0 be a given graph, $x \epsilon V(G_0)$, and G_1 a subgraph of G_0 such that $V(G_1) = V(G_0) - \{x \cup \Gamma x\}$. A set S of vertices in G_1 is a complete IS set of G_1 if and only if $S \cup \{x\}$ is a complete IS set of G_0.

2. ALGORITHM

The considerations here are based on the results and concepts of Hakimi and Frank in [1]. First we recall the concept of alternating forest.

A subforest $F = (V(F), E(F))$ of a graph G_0 is a subgraph of G_0 which contains no circuits, $V(F) \subset V(G_0)$, and $|\Gamma x| > 0$ for each vertex $x \epsilon V(F)$. In general, F consists of several separate, con-

nected subgraphs, subtrees, of G_0, but it may be a subtree as well. A vertex x of G_0 is a pendant vertex, if $|\Gamma x| \leq 1$. Further, if G_i is a subgraph of G_0, $\Gamma_i x$ is the set of the vertices adjacent to x in G_i. Following Hakimi and Frank, we shall call the vertices of an *IS* set in G_0 dark vertices and the remaining vertices light vertices. An alternating forest is a subforest of G_0 with properties given in the following definition (Definition 1 in [1]).

Definition 1. Let S be an *IS* set of a graph G_0. A subgraph $G_1 = (V(G_1), E(G_1))$ of G_0 is called an alternating forest of G_0 with respect to S, if G_1 is a subforest of G_0 satisfying the following conditions:

(i) There exists no pair of distinct vertices x and y belonging to the set $V(G_1) \cap \bar{S}$ such that $y \epsilon \Gamma x$. (No pair of light vertices in $V(G_1)$ are adjacent in G_0.)

(ii) The set of all pendant vertices of G_1, defined by $P = \{x \epsilon V(G_1) : |\Gamma_1 x| = 1\}$, is a subset of $\bar{S}$. (All pendant vertices of G_1 are light.)

(iii) $|\Gamma x \cap \{S \cap \overline{V(G_1)}\}| = 0$ for any $x \epsilon V(G_1) \cap \bar{S}$. (No light vertex in $V(G_1)$ is adjacent to a dark vertex in $\overline{V(G_1)}$.)

The two following facts, proved by Hakimi and Frank, motivate the presentation and consideration of the concept of alternating forest:

LEMMA 2: (Theorem 1 in [1]) A complete *IS* set S of a graph G_0 is a *MIS* set of G_0 if and only if there exists no alternating forest in G_0 with respect to S.

LEMMA 3: (See the proof of Theorem 1 in [1]) Let F be an alternating forest with respect to a complete *IS* set S in a graph G_0. Then $S_1 = \{S - S \cap V(F)\} \cup \{\bar{S} \cap V(F)\}$ is a complete *IS* set of G_0 such that $|S_1| > |S|$.

Thus, the finding of an F always gives a new complete *IS* set being nearer to a *MIS* set of G_0 than the original one. The following algorithm gives a way of detecting an alternating forest F with respect to a given complete *IS* set S in a graph G_0, if such exists. Hence, it operates as an algorithm for finding of a *MIS* set of a given graph G_0.

ALGORITHM: Let G_0 be a given graph and S a complete *IS* set of G_0.

(1) Remove from G_0 all the edges connecting two light vertices and denote the graph thus obtained by G_0''.

(2) Remove from G_0'' all the vertices and the edges incident to those, which form a circuit in G_0'', and denote the graph thus obtained by G_1''.

(3) Remove from G_1'' all the light and nonpendant vertices and the edges incident to those, which are adjacent to a light vertex in G_0. Denote the graph thus obtained by G_2''.

(4) Remove from G_2'' all pendant and dark vertices and the edges incident to them, and denote the graph thus obtained by G_3''.

(5) Apply (4) to G_3''. After (4) does not alter G_3'', G_3'' is an alternating forest of G_0 with respect to S, if $V(G_3'') \neq \phi$, and, if $V(G_3'') = \phi$, S is a *MIS* set of G_0.

Lemma 1 implies how one can easily find a complete *IS* set of G_0 needed in the algorithm above. Let $V(G_{i+1}) = V(G_i) - \{x_i \cup \Gamma_i x_i\}$ for $i = 0, 1, \ldots, n-1$, and $V(G_n) = \phi$, then, according to Lemma 1, $\{x_0, x_1, \ldots, x_{n-1}\}$ is a complete *IS* set of G_0.

3. SOME REMARKS AND A PROOF OF THE ALGORITHM

In this section we shall make some remarks on the applying of the algorithm and show the validity of the algorithm. At first we consider the proof.

The main idea of the algorithm is to identify the vertices not belonging to an alternative forest F with respect to a complete *IS* set of G_0. The properties (iii) and (ii) of F in Definition 1 imply that for

every pendant vertex w of F in G_0, the set $\Gamma w \cap \overline{V(F)}$ belongs to the set of light vertices. Hence, all the edges joining two light vertices in G_0 can be removed without altering the forest F. This proves (1). Let G_0'' be the graph thus obtained and consider a connected component of G_0'', say G_{0j}''. If u is a vertex belonging to S and to a circuit in G_{0j}'', $u \notin V(F)$. Indeed, according to the operation in (1), for every light vertex $v \in V(G_{0j}'')$, $\Gamma_{0j} v$ consists of dark vertices only, and then the cutting-off of the circuit gives a contradiction either to (iii) or to (ii). Thus the operation in (2) in the algorithm can be performed without altering F.

The operation (3) in the algorithm is a direct consequence of (i) in Definition 1, and according to (ii) in Definition 1, if $y \in S$ and y is a pendant vertex in G_3'', $\{\Gamma_3 y \cup y\} \cap V(F) = \phi$, and the vertices of $\{\Gamma_3 y \cup y\}$ can be removed without altering F. This proves (4).

From the operation in (2) it follows that G_3'' is a subforest of G_0, if $V(G_3'') \neq \phi$. The operations of the algorithm have removed all the vertices not belonging to an F, and hence $V(G_3'') \subset V(F)$, if an F exists. On the other hand, the proofs of (1), (2), (3), and (4) imply that $V(G_3'') \supset V(F)$, and as the operations have not altered F, $F = G_3''$, if such exists. This completes the proof of the algorithm.

In the latter part of this section we make some observations, which show a reasonable way of constructing a complete *IS* set of G_0, and the set to which it is sufficient to apply the algorithm.

The following lemma is trivial, but its partial converse, proved below, simplifies the determination of a *MIS* set in G_0.

LEMMA 4: Let S be a *MIS* set of a given graph G_0 and let $x V(G_0)$ then $S - \{x\}$ is a *MIS* set of the subgraph G_1 of G_0, where $V(G_1) = V(G_0) - \{x \cup \Gamma x\}$.

We shall say that a vertex x, $x V(G_0)$, has a complete neighborhood in G_0, if for any two vertices $z, y \in \Gamma x$, the edge (z, y) belongs to G_0.

LEMMA 5: Let G_0 be a given graph and $x V(G_0)$. If x is a pendant vertex of G_0, or if x has a complete neighborhood in G_0, then $S \cup \{x\}$ is a MIS set of G_0, if S is a *MIS* set of the subgraph G_1 of G_0, $V(G_1) = V(G_0) - \{x \cup \Gamma x\}$.

PROOF: Let x be a pendant vertex of G_0. If there is an alternating forest F in G_0, according to (ii) of Definition 1, $x \notin V(F)$, and according to (iii), $\Gamma x \notin V(F)$.

Let x be a vertex with a complete neighborhood in G_0. According to (ii) in Definition 1, x cannot be a pendant vertex of F, and (i) in Definition 1 implies that any path of an alternating forest cannot contain the vertex x. Further, if a vertex y, $y \in \Gamma x$, belongs to F, then (iii) in Definition 1 gives a contradiction.

Accordingly, if S is a *MIS* set of G_1, G_0 does not contain an alternating forest, and hence $S \cup \{x\}$ is a *MIS* set of G_0. This completes the proof.

The following lemma shows how one can simplify, by the aid of Lemma 1, the problem to find a *MIS* set of a given graph G_0.

LEMMA 6: Let G_0 be a given graph, G_1 a subgraph of G_0 such that $V(G_1) = V(G_0) - \{x \cup \Gamma x\}$, and let S be a *MIS* set of the graph G_1. Let $S'(\Gamma x)$ denote an *IS* set of the subgraph of G_0 induced by the vertex set $\{x \cup \Gamma x\}$ such that $S \cup S'(\Gamma x)$ is a complete *IS* set of G_0. $S \cup S'(\Gamma x)$ is a *MIS* set of G_0 if and only if there is no alternating forest in G_0 with respect to $S \cup S'(\Gamma x)$ containing a vertex of the set $\{x \cup \Gamma x \cup \Gamma(\Gamma x)\}$.

PROOF: If there is an alternating forest F in G_0 such that $V(F) \cap \{x \cup \Gamma x \cup \Gamma(\Gamma x)\} = \phi$, S is not a *MIS* set of G_1 which is a contradiction, and the lemma follows.

It is an exercise to construct an example, where F is an alternating forest with respect to $S \cup S'(\Gamma x)$ in G_0 and the only vertex $z \in V(F) \cap \{x \cup \Gamma x \cup \Gamma(\Gamma x)\}$ belongs to $\Gamma(\Gamma x)$ and $z \notin \{x \cup \Gamma x\}$. Thus, Lemma 6 shows where in G_0 it is reasonable to try to find an alternating forest with respect to $S \cup S'(\Gamma x)$.

4. TWO EXAMPLES

The first of our examples shows the significance of Lemma 5; an *MIS* set has been obtained directly. The same example has been solved in [2], pp. 213–218, in various Boolean and pseudo-Boolean ways, and one can easily see the power of Lemma 5. In the second example, a synthesis of Lemma 5 and the algorithm has been used.

Consider the graph of Figure 1. Vertex y has a complete neighborhood in G_0, and hence we remove $\{y \cup \Gamma x\}$ from G_0. In the graphs G_1, G_2, and G_3 (see the process illustrated in Figure 1) vertices v, w, and p, respectively, are pendant vertices, and trivially r is the *MIS* set of G_4. Thus, the application of Lemma 5 shows that $\{y, v, w, p, r\}$ is a *MIS* set of G_0.

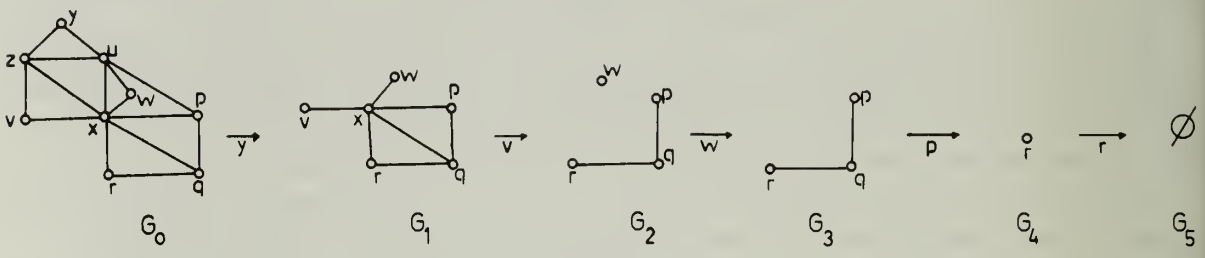

FIGURE 1

Consider the graph G_0 of Figure 2. We start the process of determining a complete *IS* set at vertex x, since no vertex of G_0 is a pendant vertex nor has a complete neighborhood in G_0. According to Lemma 5, $\{y, z, w\}$ is a *MIS* set of G_1 (see the process illustrated in Figure 2). We put $S'(\Gamma x) = \{r, s\}$, and thus the vertex set $S \cup S'(\Gamma x)$ of Lemma 6 is $\{y, z, w, r, s\}$. Figure 3 illustrates the set $S \cup S'(\Gamma x)$ in G_0. Now, we apply the algorithm to G_0 in order to see whether or not G_0 contains an alternating forest with respect to $S \cup S'(\Gamma x)$.

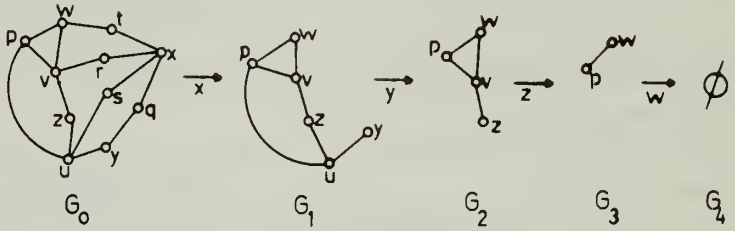

FIGURE 2

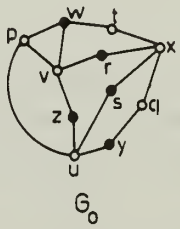

FIGURE 3

After performing the operation in (1), we obtain G_0'' of Figure 4. The application of (2) to G_0'' implies the removal of vertices v, r, x, s, u, z, and we obtain the graph G_1''. (3) does not imply any removals, and (4) requires the removal of $\{y \cup \Gamma_1'' y\} = \{y, q\}$. The graph thus obtained is G_2'' and it is an alternating forest (a tree in this case) with respect to $S \cup S'(\Gamma x)$. The new complete *IS* set of G_0 given by

$G_2'' = F$ is $(\{y, z, w, r, s\} - \{y, z, w, r, s\} \cap \{t, w, p\}) \cup (\{t, q, u, v, p\} \cap \{t, w, p\}) = \{y, z, r, s, t, p\} = N$. There is no alternating forest with respect to N in G_0, and hence N is a *MIS* set of G_0.

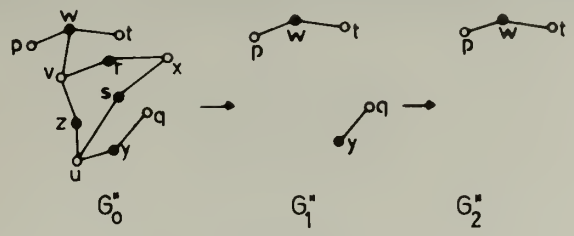

FIGURE 4

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A NOTE ON A CONSTRAINED REPLACEMENT MODEL FOR SHIPS SUBJECT TO DEGRADATIONS IN UTILITY

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ABSTRACT

In a recent paper, Peter J. Kalman considers a stochastic constrained optimal replacement model using the case of ship replacement as an example. However, the development of the constrained model proposed is not pursued and the optimal interval between replacements is determined in the absence of constraints. Among other things, it is the purpose of the present paper to extend the previous results to develop explicit types of constraints for the case of ship replacement and to determine how the optimal replacement interval may change as a result of these constraints. The constraints are concerned with the state of readiness of each ship in a group of ships. Readiness is assumed to be measured on an ordinal valued utility scale. It is proposed that ordinal valued data on ship combat readiness ratings collected by the Navy may be a useful source of empirical information for a model of the type discussed.

INTRODUCTION

The starting point of the present effort is a paper by Kalman [2] which considers the optimal replacement interval for groups of machines under certain assumptions. Although the terms "machines" and "ships" are often used interchangeably, the primary interest in this paper lies in studying problems relating to the replacement of ships.

Kalman's model calls for the minimization of the net present value of all future costs of an incumbent set of machines (ships) and their infinite chains of replacements. Although each incumbent and its replacements are permitted to have different cost characteristics, the analysis in the case of ship replacements is carried through for the case where all incumbent ships are identical. The costs considered are the present values of initial investment costs (except for the incumbents) and operating costs, decreased by the present value of the salvage value of the incumbents and their replacements.

Since the principal purpose of this paper is to consider constrained optimization in the case of ship replacement, all the assumption of [2] regarding costs, etc. will be retained. Briefly stated they are:

1. All successive ships replacing the incumbent will be retained for equal lifetimes, L. The life of the incumbent ship is denoted as L_0.
2. The operating cost of each ship does not depend on its state of readiness.
3. Only replacements are considered (In his general model Kalman considers modernizations as well as replacements).
4. A replacement can be effected instantaneously.
5. The initial investment costs, W, for all replacements are the same. Although not explicitly stated, it is also assumed that the only investment required during the life of each ship is the initial investment.

6. The salvage values, S, are the same for each ship in the chain of replacements and do not depend on L, the life of the ship when it is replaced. The salvage value of the incumbent ship is assumed to be S_0, independent of L_0.

7. The operating cost of a ship during its lifetime does not depend on the intensity of utilization of the ship.

8. The operating cost functions of the incumbent ship and its replacements are assumed to be of the form

$$\phi_0(t) = E_0 + \beta_0 t$$

$$\phi_k(t) = E - \alpha(L_o + [k-1]L) + \beta t, \qquad k = 1, 2, 3. \ . \ . \ .,$$

where

ϕ_0 is the operating cost of the incumbent ship,

ϕ_k is the operating cost of the kth replacement,

and E, E_0, β_0, α, β, are nonnegative parameters.

The above equations assume linear operating cost functions for the incumbent and replacements where the "base" operating cost rate for the kth replacement is shifted down at a rate α per time period starting from an initial level E. In general, presumably $E \leqslant E_0$ and $\alpha + \beta \geqslant 0$.

9. Interest is assumed to be compounded continuously at a nominal annual rate, r.

Under the above assumptions Kalman determines optimal values for L_0 and L. It can easily be seen that the optimal value of $L = L^*$ is independent of L_0 and can be determined from the solution of the following equation:

$$L - \frac{r(W-S)}{\alpha+\beta} = \frac{(1-e^{-rL})}{r}.$$

The optimal value of L_0 is then determined from

$$L_0^* = \frac{(E-E_0)}{\alpha+\beta_0} + \frac{r(S-S_0)}{\alpha+\beta_0} + \frac{(\alpha+\beta)L^*}{(\alpha+\beta_0)}.$$

For further details concerning these assumptions and results see Reference [2].

UTILITIES

In [2], each machine is represented by an ordinal measure of its utility characterized by one of the integers $1, 2, \ldots q$ in decreasing order of desirability. Every machine is assumed to belong to one and only one utility class at any point in time. With respect to ships, in accordance with [2] the notion of utility-worth class is interpreted as the notion of military readiness characterized by one of the integers $1, 2, \ldots q$ in decreasing order of desirability.

In practice, an ordinal measure of ship readiness is actually employed by various Navy fleet commands. For example, according to [1], the Pacific Fleet Operational Status (OPSTAT) system reports readiness grades for each ship under four resource areas: personnel, supply, equipment, and training. The grades are assigned by commanding officers as combat readiness ratings (often referred to as C-ratings) on an ordinal scale ranging from C–1 (fully ready) to C–4 (not ready). A summary value of ship readiness can be derived from the individual resource C-ratings. For example, one procedure

pursued in [1] and elsewhere is a weakest link approach where a summary C-rating for ship readiness, the "ship summary grade" is computed. This summary value is taken to be the worst of the individual resource C-ratings.

Thus, there is ample ordinal data existing for ship readiness determination provided one can assume that the type of information supplied on the ship's resources really measures the ship's readiness, broadly defined as the capability of the ship to perform missions which are assigned to it. Rather than pursue this question further, we shall merely assume for purposes of this paper that ship readiness can be determined on a one-dimensional ordinal scale of measurement. Further research into the problem of readiness measurement is undoubtedly required to determine whether such a measurement procedure is adequate.

CONSTRAINTS

It seems reasonable both from a logical and practical standpoint to impose constraints involving the expected number of ships desired in each utility class. Within the framework of the assumptions of [2], a reasonable requirement is that the expected number of ships suffering utility degradations (decreases in readiness) below some given level during the life of any replacement never be less than a given number.

Suppose for example $U_r(t)$ represents the maximum expected number of machines we desire to have exactly r degradations in utility at $t \geq 0$ time units after a group of replacements are put into service. Now, assume with [2] that only replacements are considered (no modernizations) and that all machines after the original group are replaced at intervals L time units apart. Now further assume that all replacements are made at the same level of utility. After such replacements, the level of any machine's utility class will decrease with time in accordance with some assumed and specified probability distribution. Define the following:

$p_j(t)$ = the probability that the degradation in utility of a machine t time units after it is put into service is equal to j utility classes ($j = 0, 1, 2 \ldots$).

Then $P_r(t) = \sum_{j=0}^{r} p_j(t)$ is the probability that the utility degradation of any machine is equal to or less than r classes after t time units. It would be logical to choose the $p_j(t)$ so that $P_r(t)$ for any value of r is a nonincreasing function of t, since one would expect utility (readiness) to be nonincreasing with the age of a machine (ship).

If it is further assumed that there are n units in service and that the degradation process of the machines (ships) takes place independently (this would rule out technological innovations which could affect all units simultaneously) then the probability that there are $k \leq n$ units in a degradation state j, t time units from the introduction of replacements, follows a binomial distribution. The expected number of machines suffering utility degradations equal to or less than r levels is given by $E_r(t)$, where

$$E_r(t) = nP_r(t) = n \sum_{j=0}^{r} p_j(t).$$

If the $P_j(t)$ are nonincreasing functions of t, then $E_r(t)$ is obviously nonincreasing in t. In such a case, the constraints that $E_r(t) \geq U_r(t)$ in the interval $0 \leq t \leq L$, for $r = 0, 1, \ldots, q$, can be replaced by the $q+1$ constraints

$$E_r(L) \leqslant U_r(L), \qquad r = 0, 1, 2, \ldots . q .$$

Here the quantity q is considered to be the maximum possible utility degradation that could take place in the interval L. If the poorest utility class possible for any machine at the start of service was C_u and the best possible utility state was C_l, then $q = C_u - C_l$.

Since all of the constraints involve a single variable L and since the $P_r(t)$ are nonincreasing functions of t, the rth constraint is equivalent to

$$L \leqslant L^{(r)}.$$

Taken together, the q constraints lead to a single constraint of the form

$$L \leqslant \min_{r=0}^{q} (L^{(r)}) = \tilde{L} .$$

The $L^{(r)}$ may have to be found by an approximation method.

A similar argument can be applied to utility degradation for the incumbent machines. If the degradation processes for these are the same as for replacements, we again will only keep the incumbents as long as the expected number having degradations equal to or less than some set of preassigned values is greater than a given set of fractions of those in service. This again leads to a single constraint on L_0 of the form

$$L_0 \leqslant \tilde{L}_0 .$$

Example

Let us carry the idea of the preceding section through by (1) taking the result of [2] for the present value w of all future costs for the ship replacement problem, (2) assuming a particular probability distribution for $p_j(t)$, (3) adding the constraints, (4) attempting to solve the new constrained nonlinear optimization problem, and (5) comparing the results with those for the unconstrained problem given by [2].

The result for the present value of future costs under the assumptions of [2] as given in section I of that paper is given by

$$w = f(L_0, L) = E_0 - e^{-rL_0} \left\{ E_0 + \beta_0 L_0 + \frac{\beta_0}{r} + rS_0 + \hat{w} \right\} + \frac{\beta_0}{r}$$

where

$$\hat{w} = E - \alpha L_0 + \frac{\beta}{r} + \frac{\{r(W - Se^{-rL}) - (\alpha+\beta)Le^{-rL}\}}{(1 - e^{-rL})} .$$

Let us now choose a distribution $p_j(t)$ which might be reasonable from a practical viewpoint. A not unreasonable choice is to assume that utility or readiness changes for each ship occur independently and at random. If we invoke the other usual assumptions (see [3] for example) which lead one to a Poisson process, we are led to the result that the probability that the number of utility degradations (represented by the random variable X) occurring in an interval of length, t, is equal to j and is

given by

$$\text{Prob}\{X=j\}=p_j(t)=\frac{(\lambda t)^j e^{-\lambda t}}{j!},$$

where λ = the average number of degradations per unit time. We shall make the additional initial simplifying assumption that λ does not depend on the usage rate of the ship nor on its initial and operating costs or salvage values. Such an assumption may obviously not be reasonable in many circumstances. Then

$$P_r(t)=e^{-\lambda t}\sum_{j=0}^{r}\frac{(\lambda t)^j}{j!}$$

and

$$E_r(t)=nP_r(t)\cdot$$

Note that $P_r(t)$ and $E_r(t)$ are decreasing functions of t (as can easily be seen by calculating $dP_r(t)/dt$). Note also that we are assuming an infinite number of utility classes rather than the finite number q of [2]. This is primarily a matter of mathematical convenience. Suppose now that for simplicity only two constraints are considered and that one specifies that (a) the expected number of replacement ships suffering readiness degradations ≤ 2 readiness levels during their lifetimes is at least 75 percent of the number of replacement ships in service and (b) that the expected number of original ships suffering degradations ≤ 2 levels during their lifetimes is at least 50 percent of the number of original ships in service. This gives

$$E_r(L)=ne^{-\lambda L}\left\{1+\lambda L+\frac{\lambda^2L^2}{2}\right\}\geq 0.75\ n$$

and

$$E_r(L_0)=ne^{-\lambda L_0}\left\{1+\lambda L_0+\frac{\lambda^2L_0^2}{2}\right\}\geq 0.50\ n.$$

Thus, the optimization problems becomes

(1) $$\text{Min } w=f(L_0, L),$$

subject to

$$e^{-\lambda L}\left\{1+\lambda L+\frac{\lambda^2L^2}{2}\right\}\geq 0.75$$

$$e^{-\lambda L_0}\left\{1+\lambda L_0+\frac{\lambda^2L_0^2}{2}\right\}\geq 0.50$$

$$L_0, L\geq 0.$$

We can determine how the optimal replacement interval may change as a result of the constraints by calculating the unconstrained values of L_0 and L and then noting whether these values so obtained violate the constraints. For example, in [2] with $E=E_0=30$ million dollars, $\alpha=0.03$, $\beta=\beta_0=1.25$, $\gamma=0.10$, $W=360$ million dollars, $S=S_0=1$ million dollars, the optimal unconstrained solutions are $L_0=L=37$ years.

The optimal solution to the general constrained problem of which (1) is an example is easy to describe. In general, we will have a set of constraints in L_0 which do not involve L and a set of constraints in L which do not involve L_0. As noted before, the optimal value of $L = L^*$ to the unconstrained problem determined from $\frac{\partial f(L_0, L)}{\partial L} = 0$ is independent of L_0. Thus it is merely necessary to determine whether L^* is or is not greater than the smallest value of L satisfying all of the constraints involving L, which we called $\tilde{L}$. If $L^* \leq \tilde{L}$, then the optimum replacement interval is L^*. If $L^* > \tilde{L}$, then ships must be replaced after $\tilde{L}$ time periods. Similarly, if L_0^* is the optimum life of the original set of ships for the unconstrained problem, and if $\tilde{L}_0$ is the smallest value of L_0 satisfying all of the constraints involving L_0, then $L_0^* \leq \tilde{L}_0$ implies that L_0^* is the optimal life for the constrained problem. If $L_0^* > \tilde{L}_0$, then the optimal value of L_0 is $\tilde{L}_0$. All of this follows from the fact that $f(L_0, L)$, in general, has a unique minimum at L_0^*, L^*.

In our example, if we choose $\lambda = 0.1$ degradation per month (about one degradation level per year) for illustrative purposes, it is easy to determine that $\tilde{L} \sim 17$ months and that $\tilde{L}_0 \sim 27$ months. Since the values of L^* and L_0^* given in [2] are $L^* = L_0^* = 37$ years for the case considered, it is seen that at least for this example, utility or readiness considerations are the predominant factor rather than costs, and the optimal lives of the replacements in the constrained problem are only 17 months whereas the incumbents have optimal lives of 27 months under the constraints. As noted previously, modernization, an important procedure for extending the useful lives of machines or ships is not considered either in this paper or in [2].

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